University of Zurich[UZH]

Master Thesis
Hodge Degeneration Theorem

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Abstract

We develop the background needed to understand the proof of the degeneration of Hodge to de Rham spectral sequence given in [2] and later re-exposed in [1]. The strategy is to exhibit in positive characteristic under mild assumptions a decomposition of the de Rham complex. The existence of this decomposition immediately implies the degeneration. Then one can lift the result to characteristic 0. The same decomposition with the same strategy also allows one to prove with very little extra work the Kodaira-Akizuki-Nakano vanishing theorem for smooth projective varieties.

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Contents

1 Preliminaries

In this section, we recall some basic notions that will be extensively used later on. The notation introduced here will be used in the rest of the text. No attempt at being comprehensive is made.

1.1 Homological algebra

We follow [4] and [10].

Throughout this section $\mathcal{A}$ will denote an abelian category. All functors between abelian categories will be assumed additive. We will denote by $C(\mathcal{A})$ the category of complexes in $\mathcal{A}$ with differential of degree 1 and $K(\mathcal{A})$ the category obtained by identifying homotopic morphisms in $C(\mathcal{A})$. If $\mathcal{A}$ is the category of $\mathcal{O}_X$-modules for a scheme X we will write $C(X)$ instead of $C(\mathcal{A})$. We denote by $C^+(\mathcal{A})$ (respectively $C^-(\mathcal{A})$, $C^b(\mathcal{A})$) the full subcategory of complexes *bounded below* (respectively *bounded above*, *bounded*) meaning the entries are zero for all indices small enough (respectively large enough, outside a finite interval).

We will often regard an object A of $\mathcal{A}$ as an object of $C(\mathcal{A})$ by identifying it with the complex with A located in degree 0 and 0 on non-zero degrees. For a complex K in $C(\mathcal{A})$, we denote by $\tau_{\leq n}K$ the sub complex given by:

$$(\tau_{\leq n}K)^i = \begin{cases} K^i & i < n \\ Z^iK & i = n \\ 0 & i > n \end{cases}$$

We use $\tau_{<n}$ as shorthand for $\tau_{\leq n-1}$.

One problem with $K(\mathcal{A})$ is that it is no longer abelian unlike $C(\mathcal{A})$. Therefore on the surface we lose our short exact sequences which give rise to long exact sequences in homology. However, we will see that we can replace them with the so called distinguished triangles.

Definition 1.1.1. Given a complex K, we can define the shifted complex by: $K[n]^i = K^{i+n}$ and differential $d^i_{K[n]} = (-1)^n d^{i+n}_K$. Clearly this defines a functor, with the obvious action on the morphisms.

Definition 1.1.2. A triangle in $K(\mathcal{A})$ is a sequence of maps of the form

$$A \to B \to C \to A[1].$$

Certain triangles called exact triangles give rise to long exact sequences of cohomology groups. These are the triangles admitting a diagram like below

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & A[1] \arrow[d] \\ A' \arrow[r, "u"] & B' \arrow[r] & cone(u) \arrow[r, "\delta"] & A[1], \end{tikzcd}$$

Where vertical maps are homotopy equivalences, and δ is the obvious projection morphism. In the following section, we will discuss the derived category of $\mathcal{A}$, denoted by $D(\mathcal{A})$. $D(\mathcal{A})$ inherits a triangulated category structure from $K(\mathcal{A})$. Meaning it also has the shift functor and a collection of exact triangles satisfying the axioms of triangulated categories. Given a short exact sequence in $C(\mathcal{A})$, one can always find a corresponding exact triangle $D(\mathcal{A})$. In this way, these triangles replace short exact sequences.

1.2 Derived categories

Given a category $\mathcal{C}$ and a collection of morphisms S in it, one can construct $\mathcal{C}[S^{-1}]$ the localization of $\mathcal{C}$ at S obtained by formally inverting all morphisms in S. This means it is the universal category which admits a functor $F : \mathcal{C} \to \mathcal{C}[S^{-1}]$ such that $F(s)$ is an isomorphism for $s \in S$. This means given $G : \mathcal{C} \to \mathcal{D}$ a functor such that $G(s)$ is an isomorphism for all $s \in S$ then we get a diagram as follows:

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[dr, "G"'] & \mathcal{C}[S^{-1}] \arrow[d, dashed] \\ & \mathcal{D}. \end{tikzcd}$$

We will be interested in localizing $K(\mathcal{A})$ at quasi-isomorphisms since we are interested usually in the cohomology of a complex rather than the complex itself. However the problem we face is the fact that describing the arrows of this localized category is not an easy task at all. It is possible under certain conditions, such as when one has extra structure on the collection S. For instance when viewing a ring as an additive category with a single object, the localization of it on a multiplicative set of morphisms coincides with the usual localization of rings. In this case there is a sort of calculus of fractions one can build. This motivates our next definition

Definition 1.2.1. A class of morphisms S in a category $\mathcal{C}$ is said to be localizing if the following hold

- S is closed under compositions and includes the identities of all objects.
- *Extension conditions:* For f any morphism and $s \in S$, there exist morphisms g (not necessarily in S) and $t \in S$ such that the square below commutes

$$\begin{tikzcd} W \arrow[r, dashed, "g"] \arrow[d, dashed, "t"'] & Z \arrow[d, "s"'] \\ X \arrow[r, "f"] & Y \end{tikzcd} \quad \left(resp. \begin{tikzcd} W & Z \arrow[l, dashed, "g"] \\ X \arrow[u, dashed, "t"'] & Y \arrow[l, "f"] \arrow[u, "s"'] \end{tikzcd} \right)$$

- If $f, g : X \to Y$ two morphisms. There exists $s \in S$ with $sf = sg$ if and only of there exists $t \in S$ with $ft = gt$.

If $\mathcal{C}$ is a category and S a localizing set of morphisms, define $\mathrm{Hom}_{\mathcal{C}[S^{-1}]}(X, Y)$ as the set of equivalence classes of diagrams of the form

$$\begin{tikzcd} & X' \arrow[dl, "s"'] \arrow[dr, "f"] & \\ X & & Y \end{tikzcd}$$

where $s \in S$ and f any morphism. Two such diagrams are equivalent if there exists a diagram of the form

$$\begin{tikzcd} & X''' \arrow[dl, "r"'] \arrow[dr, "h"] & \\ X' \arrow[d, "s"'] \arrow[drr, "f", near start] & & X'' \arrow[d, "g"] \arrow[dll, "t"', near start] \\ X & & Y \end{tikzcd}$$

with $sr \in S$. The structure of S makes sure this is a well defined equivalence relation. Composition will be given by

$$\mathrm{Hom}_{\mathcal{C}[S^{-1}]}(X, Y) \times \mathrm{Hom}_{\mathcal{C}[S^{-1}]}(Y, Z) \to \mathrm{Hom}_{\mathcal{C}[S^{-1}]}(X, Z)$$

$$
\begin{tikzcd}[column sep=small]
 & X' \arrow[dl, "s"'] \arrow[dr, "f"] & & & & Y' \arrow[dl, "t"'] \arrow[dr, "g"] & \\
X & & Y & \times & Y & & Z
\end{tikzcd}
\quad \mapsto \quad
\begin{tikzcd}[column sep=small]
 & & X'' \arrow[dl, dashed, "t'"'] \arrow[dr, dashed, "h"] & & \\
 & X' \arrow[dl, "s"'] \arrow[dr, "f"] & & Y' \arrow[dl, "t"'] \arrow[dr, "g"] & \\
X & & Y & & Z.
\end{tikzcd}
$$

Existence of which is guaranteed by the extension axiom. The collection of quasi-isomorphisms in the categories $K(\mathcal{A}), K^+(\mathcal{A}), K^-(\mathcal{A})$ are localizing. Denote by $D^\bullet(\mathcal{A})$ the localization of $K^\bullet(\mathcal{A})$ at quasi-isomorphisms. It is clear what the $\bullet$ can stand for. We will denote by $q_{\mathcal{A}}$ the natural functor $q_{\mathcal{A}} : K^\bullet(\mathcal{A}) \to D^\bullet(\mathcal{A})$.

Definition 1.2.2. Given a left exact functor between abelian categories $F : \mathcal{A} \to \mathcal{B}$, it trivially induces a morphism between $F : K(\mathcal{A}) \to K(\mathcal{B})$ (by abuse of notation, they will be called by the same letter. Context should make it clear which one is being talked about). Define $RF : D(\mathcal{A}) \to D(\mathcal{B})$ to be the right Kan extension of $q_{\mathcal{B}} \circ F$ along $q_{\mathcal{A}}$. This is called the right derived functor of F. Dually, if F was a right exact functor we define $LF : D(\mathcal{A}) \to D(\mathcal{B})$ as the left Kan extension instead. This is called the left derived functor of F.

Now the natural question to ask is whether they exist. Under some hypothesis' the answer is yes. Here, we will treat the case of a left exact functor F. By replacing the statements by their duals, the case of a right exact functor can also be obtained.

The following is a crucial proposition

Proposition 1.2.3. *Let $I^\bullet$ be an object of $K^+(\mathcal{A})$ with each component injective. Let $L^\bullet$ be an element of $K(\mathcal{A})$. Then the natural map*

$$\mathrm{Hom}_{K(\mathcal{A})}(L^\bullet, I^\bullet) \to \mathrm{Hom}_{D(\mathcal{A})}(L^\bullet, I^\bullet)$$

is an isomorphism.

Proof. We will need a lemma

Lemma 1.2.4. *Take $I^\bullet, L^\bullet$ as above. If $K^\bullet$ is another object of $K(\mathcal{A})$ with a quasi-isomorphism $\alpha : K^\bullet \to L^\bullet$, there exists a unique $\beta \in \mathrm{Hom}_{K(\mathcal{A})}(L^\bullet, I^\bullet)$ such that the diagram below commutes*

$$
\begin{tikzcd}
K^\bullet \arrow[r, "\alpha"] \arrow[d, "\gamma"] & L^\bullet \arrow[dl, dotted, "\beta"] \\
I^\bullet &
\end{tikzcd}
$$

Proof. We have a distinguished triangle

$$K^\bullet \xrightarrow{\alpha} L^\bullet \longrightarrow \mathrm{Cone}(\alpha)^\bullet \longrightarrow K^\bullet[1].$$

Since contravariant Hom is cohomological (see [Stacks, Tag 0149]) this induces an exact sequence

$$\mathrm{Hom}_{K(\mathcal{A})}(\mathrm{Cone}(\alpha)^\bullet, I^\bullet) \to \mathrm{Hom}_{K(\mathcal{A})}(L^\bullet, I^\bullet) \to \mathrm{Hom}_{K(\mathcal{A})}(K^\bullet, I^\bullet) \to \mathrm{Hom}_{K(\mathcal{A})}(\mathrm{Cone}(\alpha)^\bullet, I^\bullet).$$

Since α is a quasi-isomorphism the cone is acyclic. As $I^\bullet$ is bounded below, the leftmost and rightmost Hom sets are 0. This is due to the fact morphisms of complexes between an acyclic complex $E^\bullet$ and a complex of injectives $I^\bullet$ are homotopy equivalent if they are sent to the same morphism by the functor $\tau_{<n}$ for some n. For details, see the proof of [4] III.2.21. Then, β is the element corresponding to γ $\square$

We are almost done since a morphism in $\mathrm{Hom}_{D(\mathcal{A})}(L^\bullet, I^\bullet)$ can be represented by a roof just like in the lemma above. Utilizing the lemma we are done since such a diagram corresponds uniquely to a morphism $\mathrm{Hom}_{K(\mathcal{A})}(L^\bullet, I^\bullet)$.

□

One of the implications is, localizing the full subcategory of bounded below complexes of injectives at quasi-isomorphisms does nothing. All quasi-isomorphisms are already homotopy equivalences.

Given an abelian category with enough injectives and a bounded below complex, there exist a way of extracting a bounded below injective complex quasi-isomorphic to it. Namely given $K^\bullet$ bounded below, one can construct a bicomplex $L^\bullet$, columns of which are injective resolutions of K^i. Then the obvious morphism $K \to \mathrm{Tot}(L)$ is a quasi-isomorphism This construction is called the Cartan-Eilenberg resolution and it is functorial (i.e given a map between complexes in $K^+(\mathcal{A})$ there is a corresponding map between the resolutions and this correspondance is functorial). Note that for the existence and functoriality it is important that the columns are injective objects but the quasi-isomorphism follows from the fact that when columns are exact and the double complex lies in the second quadrant, the first spectral sequence of the total complex is concentrated in the zeroth row. For more details see [4] Lemma III.7.12.

Cartan-Eilenberg resolutions (viewed as a functor from $D^+(\mathcal{A})$ to the full subcategory of complexes with injective components) then has a natural transformation to the identity functor of bounded below injectives and $D^+(\mathcal{A})$ where (crucially) each component of the transformation is a quasi-isomorphism. This induces and equivalence between the derived categories, since after passing to them the natural transformation becomes an isomorphism. Then given left exact F we can calculate its right derived functor RF using this equivalence of categories. Moreover, since quasi-isomorphisms of injectives correspond (uniquely) to homotopy equivalences, they are taken to homotopy equivalences so RF is even an exact functor of triangulated categories since it transforms exact triangles into exact triangles.

Remark 1.2.5. K-Injective complexes are defined as complexes satisfying the crucial proposition above without being bounded below. After showing their existence and that one can functorially embed any complex into them they can be used to construct derived functors on unbounded complexes as well. It turns out every Grothendieck abelian category has functorial resolutions by K-Injective complexes. In the rest of the text, all applications involve bounded below or above complexes. However, we will take the liberty to use notation as if the derived complexes are defined everywhere. For explicit constructions see [Stacks, Tag 01D4].

Definition 1.2.6. An object K of $D^b(A)$ is called decomposable if K is isomorphic in $D(A)$ to a complex with zero differential.

If L is decomposable and $u : K' \to K$ is the isomorphism with the complex with zero differentials K', we have isomorphisms $K'^i \xrightarrow{\sim} H^iK' \xrightarrow{\sim} H^iK$. Therefore in $D(A)$

$$K \cong \bigoplus (H^iK)[-i]$$

and clearly, if above is satisfied then K is decomposable. A choice of isomorphism inducing the identity on H^i for all i is called a *decomposition* of K.

Suppose $T : \mathcal{A} \to \mathcal{B}$ is a left exact functor between abelian categories and $\mathcal{A}$ has enough injectives. Then there exist a right derived functor $RT : D^+(\mathcal{A}) \to D^+(\mathcal{B})$ of it as discussed above. The i-th cohomology of $RT(K)$ for a complex K is denoted by $R^iT(K)$. It can be computed as follows:

Recall the fact that when columns of a double complex are exact, the first spectral sequence of the total complex is concentrated in the zeroth row. For more details see [4] Lemma III.7.12. If we apply the functor T termwise to the Cartan-Eilenberg resolution, the columns stay exact as injectives are acyclic for all left exact functors. Then the first spectral sequence of the double complex gives us

$$E_1^{ij} = R^jT(K^i) \Rightarrow R^*T(K).$$

This is usually called the first spectral sequence of hypercohomology of T.

1.3 Simplicial objects and standard resolution

Our objective is to define the standard resolution with respect to a pair of adjoint functors. Later, we will exploit Dold-Kan correspondence to use this standard resolution to define the cotangent complex of a ring map. We follow [Stacks, Tag 0162]

Let Δ denote the category with

$$\text{Objects: Sets } [n] = \{0, 1, 2, \cdots, n\} \text{ for each } n \in \mathbb{N}$$
$$\text{Morphisms: } f : [n] \to [m] \text{ nondecreasing map of sets}$$

There are 2 types of particularly important morphisms. These are, for $n \geq 0$ and $0 \leq j \leq n$, $\sigma_j^n : [n+1] \to [n]$ the surjective order preserving map with $(\sigma_j^n)^{-1}(\{j\}) = \{j, j+1\}$. For $n \geq 1$ and $0 \leq j \leq n$, $\delta_j^n : [n-1] \to [n]$ the order preserving map which skips j. Any morpism in this category is a composite of the above two types of maps. Furthermore the relations below can easily be observed to be satisfied.

(a) If $0 \leq i < j \leq n+1$, then $\delta_j^{n+1} \circ \delta_i^n = \delta_i^{n+1} \circ \delta_{j-1}^n$.

(b) If $0 \leq i < j \leq n-1$, then $\sigma_j^{n-1} \circ \delta_i^n = \delta_i^{n-1} \circ \sigma_{j-1}^{n-2}$.

(c) If $0 \leq j \leq n-1$, then $\sigma_j^{n-1} \circ \delta_j^n = \mathrm{id}_{[n-1]}$ and $\sigma_j^{n-1} \circ \delta_{j+1}^n = \mathrm{id}_{[n-1]}$.

(d) If $0 < j+1 < i \leq n$, then $\sigma_j^{n-1} \circ \delta_i^n = \delta_{i-1}^{n-1} \circ \sigma_j^{n-2}$.

(e) If $0 \leq i \leq j \leq n-1$, then $\sigma_j^{n-1} \circ \sigma_i^n = \sigma_i^{n-1} \circ \sigma_{j+1}^n$.

Since any morphism can be factored into these two types of maps, and the above relations exhaust all possible combinations of them on a square diagram. Therefore above data determine a presentation of the category Δ.

Definition 1.3.1. A *simplicial object* in a category $\mathcal{C}$ is a presheaf on Δ valued in $\mathcal{C}$. In other words a functor $\Delta^{op} \to \mathcal{C}$

Given such a functor F, one can also interpret it as a set of objects $X_n \in \mathcal{C}$ with $X_n = F([n])$ and a collection of morphisms $d_j^n = F(\delta_j^n) : X_n \to X_{n-1}$ and $s_j^n = F(\sigma_j^n) : X_n \to X_{n+1}$ such that these morphisms satisfy the opposite of the identities mentioned above. These will be called the simplicial identities. We will call d_n^j the j-th degeneracy map and s_n^j the j-th face map (we will often drop n from notation when the source and the target are clear). Clearly, a morphism of simplical objects is a natural transformation of functors, so a map between each object of the collection which commutes with face and degeneracy maps.

Conversely in a category $\mathcal{C}$ a collection of objects X_n indexed by $\mathbb{N}$ together with face and degeneracy maps satisfying the simplicial identities determines a functor Δ^{op} by the virtue of the fact that Δ admits a presentation by precisely opposites of the simplicial identities. Below are some examples

Example 1.3.2. Given an object $X \in \mathcal{C}$ we can construct the constant simplicial object $\underline{X}$ given by $X_n = X$ for all n where the face and degeneracy maps are id_X

Example 1.3.3. The functors $\mathrm{Hom}_\Delta(-, [n])$ determine simplicial objects in the category of sets. These will be referred to as simplicial sets. We will denote them by $\Delta[n]$. These are particularly important since, given another simplicial set X the Yoneda lemma tells us

$$X_n = X([n]) = \mathrm{Hom}_{Psh(\Delta)}(\Delta[n], X)$$

The following example is a bit more complicated, however it will be the one most relevant to us later.

Example 1.3.4. Given adjoint pair of functors $U \dashv V$, $U : \mathcal{S} \to \mathcal{A}$ we have two natural transformations

$$d : U \circ V \to \mathrm{id}_{\mathcal{A}} \quad \nu : \mathrm{id}_{\mathcal{S}} \to V \circ U$$

namely the unit and the counit of the adjuntion. Out of these we can construct a simplicial object in the category $\mathrm{Fun}(\mathcal{A}, \mathcal{A})$. We let

$$X_n = (U \circ V)^{\circ(n+1)}$$
$$d_j^n = \mathrm{id}_{X_{j-1}} \star d \star \mathrm{id}_{X_{n-j-1}} \quad s_j^n = \mathrm{id}_{X_{j-1}} \star \nu \star \mathrm{id}_{X_{n-j-1}}$$

Here $\star$ denotes horizontal composition. For the details of checking that the simplicial identities are satisfied see [Stacks, Tag 08NC]

We will denote by $\mathrm{Simp}(\mathcal{A})$ the category of simplicial objects in a category $\mathcal{A}$. The following remarkable theorem tells us that when $\mathcal{A}$ is abelian, $\mathrm{Simp}(\mathcal{A})$ is equivalent to $\mathrm{Ch}_{\geq 0}(\mathcal{A})$. With appropriate model structures on both categories, this equivalence even preserves fibrations, cofibrations and weak equivalences. Even though this fact is fundamentally what we are going to exploit, we will avoid going into a discussion of model categories.

Theorem 1.3.5 (Dold-Kan)**.** *Given an abelian category $\mathcal{A}$ let $Ch_{\geq 0}(\mathcal{A})$ denote the category of $\mathcal{A}$ complexes with a differential of degree -1, with only non zero entries in degrees ≥ 0. Then there exists functors*

$$N : \mathrm{Simp}(\mathcal{A}) \leftrightarrows \mathrm{Ch}_{\geq 0}(\mathcal{A}) : S$$

which are quasi-inverse equivalences of categories.

Proof. See [Stacks, Tag 019D] □

Furthermore, below we will mention a notion of homotopies between simplicial objects (and maps). This correspondence respects homotopies, in the sense two maps will be homotopic if and only if the corresponding maps of chain complexes are homotopic.

Definition 1.3.6. Let $\mathcal{C}$ be a category where finite coproducts exist. Let U be a simplicial set. Let V be a simplicial object in C. If each U_n is finite we define the tensor product $U \otimes V$ as the simplicial object

$$(U \otimes V)_n = \coprod_{u \in U_n} V_n$$

and for $\phi : [m] \to [n]$, $(U \otimes V)(\phi)$ is given by the morphism

$$\coprod_{u \in U_n} V_n \to \coprod_{u' \in U_m} V_m$$

mapping component V_n corresponding to u to the component V_m corresponding to $U(\phi)(u)$ with the morphism $V(\phi)$.

Clearly, it can happen that although the category lacks all coproducts, if the coproducts involved in our construction exists we can still define it. Now for X an object in a category $\mathcal{C}$ for which finite coproducts with itself exist, we will denote by $X \otimes \Delta[k]$ the tensor product where X is regarded as the constant simplicial object.

Lemma 1.3.7. *With X as above, for any simplicial object V of $\mathcal{C}$ we have*

$$\mathrm{Hom}_{Simp(\mathcal{C})}(X \otimes \Delta[k], V) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(X, V_k)$$

Proof. A morphism $\varphi : X \otimes \Delta[k] \to V$ is given by a family of morphisms $\varphi_\alpha : X \to V_n$ for each $\alpha : [n] \to [k]$. Since it is supposed to be a natural transformation of functors they are supposed to satisfy the following for any $\phi : [m] \to [n]$

$$\begin{array}{ccc} X & \xrightarrow{\varphi_\alpha} & V_n \\ \big\downarrow{\scriptstyle \mathrm{id}_X} & & \big\downarrow{\scriptstyle V(\phi)} \\ X & \xrightarrow{\varphi_{\alpha \circ \phi}} & V_m \end{array}$$

Thus we see that φ_ϕ is determined by $\varphi_{\mathrm{id}_{[k]}}$. We will map φ to $\varphi_{\mathrm{id}_{[k]}}$

Conversely, given a morphism $f : X \to V_k$ we can set $\varphi_\alpha = V(\alpha) \circ f$. It is easy to see these maps are inverses of each other. □

There are two morphisms

$$e_0, e_1 : \Delta[0] \to \Delta[1]$$

induced by the two morphisms $[0] \to [1]$. Since $\Delta[0]_k = \{*\}$ is the singleton it is easy to see that given a simplicial object U in a category $\mathcal{C}$ with finite coproducts that $U \otimes \Delta[0] = U$. Then the morphisms e_0, e_1 induce

$$e_0, e_1 : U \to U \otimes \Delta[1]$$

Definition 1.3.8. Let $\mathcal{C}$ be a category with finite coproducts and U, V two simplicial objects of $\mathcal{C}$. If $a, b : U \to V$ two morphisms then

(1) We say that a morphism

$$h : U \otimes \Delta[1] \to V$$

is a *homotopy* from a to b if $a = h \circ e_0$ and $b = h \circ e_1$.

(2) We say a and b are *homotopic* if there exists a finite sequence of morphisms $a_0, ..., a_n$ with $a_0 = a$ and $a_n = b$ such that there is a homotopy between consecutive morphisms in either direction.

Being homotopic is an equivalence relation. It is precisely the equivalence relation generated by the relation given by the existence of a homotopy from a to b. Also note that one can componentwise define a homotopy by noting the identities the collection of maps should satisfy. This allows one to define homotopy in a category without the appropriate coproducts to form the product. As we have no need for this, we will not discuss it.

Example 1.3.9. In the situation of 1.3.4, in other words when constructing a simplicial object in the category of functors using an adjoint pair U, V the maps

$$\mathrm{id}_V \star \epsilon : V \circ X \to V, \quad \epsilon \star \mathrm{id}_U : X \circ U \to U$$

are homotopy equivalences. For the proof we refer to [Stacks, Tag 08P1].

We will conclude the discussion about homotopies by noting once again that, if two morphisms between simplicial objects in an abelian category are homotopic, so are the morphisms corresponding to them in the category of chain complexes. The converse is also true. These are proved in [Stacks, Tag 01A1] and [Stacks, Tag 019Q].

1.4 Sites and ringed topoi

Most proofs in this section will be left to references. A good reference for sheaves and topos theory is [9].

In this subsection, let $\mathbf{C}$ denote a category with pullbacks.

Definition 1.4.1. A (basis for a) Grothendieck topology is an assignment to each object C of $\mathbf{C}$, a collection of families of morphisms $Cov(C)$ with codomain C satisfying the following:

(a) All isomorphisms with codomain C are in $Cov(C)$

(b) Given a family $\{f_i\} \in Cov(C)$ and a morphism $d : D \to C$ then the family $\{d \times_C f_i\} \in Cov(D)$

(c) Given a family $\{f_i : C_i \to C\} \in Cov(C)$ and for each i, families $\{f_{ij} | j \in J\} \in Cov(C_i)$ then the family $\{f_i \circ f_{ij}\} \in Cov(C)$

Example 1.4.2. The example that motivates this definition is the category obtained from a topological space in the following way: Let objects be open subsets of a given topological space X and morphisms the inclusions between the open sets. Clearly, this category is closed under pullbacks, they correspond to intersections and open sets are closed under finite intersections. For U an object, let $Cov(U)$ be families of morphims $\{f_i : U_i \to U \| i \in I\}$ such that $\bigcup_{i \in I} U_i = U$. It is instructive to work out what the above given axioms correspond to in this example.

Example 1.4.3. There is a topology we can put on any category $\mathbf{C}$ where coverings for each object are $Cov(U) := \{f : V \to U \mid f \text{ isomorphism}\}$. This topology will be called the *chaotic topology* on $\mathbf{C}$.

When we have the structure of a site on $\mathbf{C}$, we can talk about what it means to for a preasheaf on $\mathbf{C}$ to be a sheaf. The definition is precisely the same with the usual topological space viewed as a site like above example.

Definition 1.4.4. Let $\mathbf{C}$ be a site and $\mathcal{F}$ a presheaf of sets on $\mathbf{C}$, F is called a sheaf if for every object U of $\mathbf{C}$ and every covering $\{U_i | i \in I\} \in Cov(U)$ the diagram below is an equalizer diagram of sets

$$\mathcal{F}(U) \longrightarrow \prod_{i\in I} \mathcal{F}(U_i) \overset{\mathcal{F}(p_1)}{\underset{\mathcal{F}(p_2)}{\rightrightarrows}} \prod_{i,j\in I} \mathcal{F}(U_i \times_U U_j)$$

Example 1.4.5. It is easy to see that for the chaotic topology on $\mathbf{C}$, all presheaves are sheaves.

Proposition 1.4.6. *There exist a fully faithful inclusion functor $i : Sh(\mathbf{C}) \to PSh(\mathbf{C})$ which is right adjoint to a functor $(\bullet)^\# : PSh(\mathbf{C}) \to Sh(\mathbf{C})$ called sheafification.*

One can find the construction in the reference mentioned in the beginning of the section. We will only mention that the sections of the sheafification functor are constructed using colimits over directed sets.

Given a functor $u : \mathbf{C} \to \mathbf{D}$ between two categories, we have several induced functors between the corresponding presheaf categories. The easiest to define is

$$u^p : PSh(\mathbf{D}) \to PSh(\mathbf{C}), \quad \mathcal{F} \mapsto \mathcal{F} \circ u$$

Note that since taking sections is exact for presheaves, this functor is exact.

There exists a left adjoint to this and it can be constructed in the following manner:

Denote by I_V the category formed by intersecting the under category of V with image of U. Given $\mathcal{G} \in PSh(\mathbf{C})$ we have a functor $\mathcal{G}_V : F_V^{op} \to Sets$ given by $\mathcal{G}_V(V \to u(U)) = \mathcal{G}(U)$ and for the morphisms similarly. Define

$$u_p\mathcal{G}(V) := \operatorname{colim}_{I_V^{op}} \mathcal{G}_V.$$

It can easily be verified that this defines a functor left adjoint to u^p. For more details, see [Stacks, Tag 00VC]. It is clear that this construction works not only with presheaves of sets but any presheaf with values in a category where the colimits we care about exist. Unfolding the definition this is the functor which takes G to its left Kan extension along u.

Similarly we can find a right adjoint to u^p using right Kan extension which is constructed similarly but using limits instead of colimits. It will be denoted ${}_pu$. Consult the same reference for more details.

Given topologies on $\mathbf{C}, \mathbf{D}$ we can try to see to what extent this functoriality between presheaves is compatible with sheaves. This leads us to the following definitions

Definition 1.4.7. Let $\mathbf{C}, \mathbf{D}$ be sites. A functor $u : \mathbf{C} \to \mathbf{D}$ is said to be *continuous* if for every $\{V_i \to V\}$ in $Cov(V)$

- $\{u(V_i) \to u(V)\} \in Cov(u(V))$.
- for any morphism $T \to V$ in $\mathbf{C}$ the morphism $u(T \times_V V_i) \to u(T) \times_{u(V)} u(V_i)$ is an isomorphism.

It is called *cocontinuous* if for every object V of $\mathbf{C}$ and every covering $\{U_i \to u(V)\}$ in $Cov(u(V))$ there exist a covering $\{V_i \to V\}$ in $Cov(V)$ such that each map in the family $\{u(V_i) \to u(V)\}$ factors through some map of $\{U_i \to u(V)\}$. Beware that $\{u(V_i) \to u(V)\}$ is not necessarily a covering in D.

Example 1.4.8. The prime example (and the reason for the name) of continuous functors between sites are those induced by a continuous function $f : X \to Y$ of topological spaces. This induces a continuous functor $Preimage(f) : Op(Y) \to Op(X)$.

For cocontinuous functors, an example is the functor induced by inclusion i of an open subset U of a topological X. $Op(U) \to Op(X)$ however, we suggest not using this for intuition, because it is quite a special case. The functor is also continuous and moreover it is fully faithful. Nevertheless it is the easiest example of a cocontinuous functor.

When the functor is continuous, it is quite easy to see that u^p takes sheaves to sheaves. Similarly, when the functor is cocontinuous, ${}_pu$ takes sheaves to sheaves. When restricted to the category of sheaves, we will repleace the p in the notation with an s.

Definition 1.4.9. Given a site $\mathbf{C}$ we will call the category $Sh(\mathbf{C})$ its associated topos. Given two sites $\mathbf{C}, \mathbf{D}$ a morphism f between their associated topoi is a pair of adjoint functors $f^{-1} \dashv f_*$ where the left adjoint is exact.

Proposition 1.4.10. *Given a cocontinuous functor $u : C \to D$ define $f^{-1} := (\bullet)^{\#} \circ u^p \circ i$ and $f_* :=_s u$ gives us a morphism of topoi $f : Sh(C) \to Sh(D)$.*

Proof. We need to show adjointness and that f^{-1} is exact. For $\mathcal{G}$ a sheaf on $\mathbf{D}$ and $\mathcal{F}$ a sheaf on $\mathbf{C}$ we have:

$$\begin{aligned} \mathrm{Hom}_{Sh(\mathbf{C})}((u^p\mathcal{G})^{\#}, F) &= \mathrm{Hom}_{PSh(\mathbf{C})}(u^p\mathcal{G}, \mathcal{F}) \\ &= \mathrm{Hom}_{PSh(\mathbf{D})}(\mathcal{G}, {}_pu\mathcal{F}) \\ &= \mathrm{Hom}_{Sh(\mathbf{D})}(\mathcal{G}, {}_su\mathcal{F}). \end{aligned}$$

Shows the adjointness. f^{-1} is right exact because it is a left adjoint, moreover it is left exact because it is a composition of left exact functors showing it is exact. ☐

Given a continuous functor $u : \mathbf{C} \to \mathbf{D}$ satisfying mild conditions (namely, that the categories I_V mentioned above are disjoint union of filtered categories) also induces a morphism of topoi $f : Sh(\mathbf{D}) \to Sh(\mathbf{C})$ given by $f^{-1} := (u_p)^{\#}$ and $f_* := u^s$. It can be proved similarly. The reason we ask for the extra condition is to make sure u_p is left exact. Note that this is not so unusual. For instance, it holds when $\mathbf{C}$ has a final object and u is left exact which is the case in the primal example of a continous functor between topological spaces.

Consider $u : \mathbf{C} \to \mathbf{D}$ is a continuous and cocontinuous functor between two sites. By above discussion there exist an adjoint triple between their categories of presheaves:

$$u_p \dashv u^p \dashv {}_pu.$$

Moreover, we have seen since u is continuous $u^p = u^s$ and since it is cocontinuous we have ${}_pu = {}_su$. Rephrasing this in terms of $f : Sh(C) \to Sh(D)$ the induced morphism between their topoi and calling $u_p := f_!$ we have

$$f_! \dashv f^{-1} \dashv f_*$$

Notice that in good situations where $f_!$ is exact, we have another morphism of topoi $g : Sh(D) \to Sh(C)$ with $g^{-1} := f_!$, $g_* := f^{-1}$.

Now we will talk about sheaves and presheaves valued not in sets but in a category of algebraic structures admitting a forgetful functor to category of sets. Namely

Proposition 1.4.11. *Consider a pair (A, s) of category A and a functor $s : A \to Sets$ satisfying*

1) s is faithful

2) A has limits and s is left exact

3) A has filtered colimits and s commutes with them

4) s reflects isomorphisms

Let $\mathbf{C}$ be a site. Denote by $PSh(\mathbf{C}, A)$ and $Sh(\mathbf{C}, A)$ the presheaves and sheaves valued in A. They satisfy the following:
(α) *Sheafification of a presheaf is a sheaf*
(β) *The diagram below commutes.*

$$\begin{array}{ccc} Sh(C,A) & \underset{(\bullet)^{\#}}{\overset{i}{\rightleftarrows}} & PSh(C,A) \\ \downarrow s & & \downarrow s \\ Sh(C) & \underset{(\bullet)^{\#}}{\overset{i}{\rightleftarrows}} & PSh(C). \end{array}$$

(δ) *$\mathcal{F} = \mathcal{F}^{\#}$ if and only if F is a sheaf.*
(ϵ) *If $\mathcal{F}$ is a sheaf, $Mor_{PSh(\mathbf{C},A)}(\mathcal{G}, \mathcal{F}) \xrightarrow{\sim} Mor_{Sh(\mathbf{C},A)}(\mathcal{G}^{\#}, \mathcal{F})$.*
(λ) *$\mathcal{F} \to \mathcal{F}^{\#}$ is left exact.*

Proof. (1,2,3,4) collectively imply that filtered colimits commute with limits in A. This is because they do in Set and those properties can easily be used to reduce the question to that case. Rest of the claims follow from the fact that, colimits involved in the construction of the sheafification functor are filtered.

(1, 2, 4) imply a presheaf on A is a sheaf if and only if the underlying set is a sheaf. This implies δ. The rest is formal. Once δ is known ϵ readily follows by the same proof as the case of sheaves valued in sets. Moreover, in good situtations like the case of a continuous functor mentioned above, (3) implies that u_p as defined here coincides with the one that is defined for underlying sets. ☐

Example 1.4.12. The category of abelian groups Ab with the forgetful functor.

Given a site $\mathbf{C}$, the category of abelian group objects in $PSh(\mathbf{C})$ is equivalent to the category $PSh(\mathbf{C}, Ab)$. This follows directly from the fact that taking sections commutes with limits. Moreover the same holds for $Sh(\mathbf{C})$ since taking sections commutes with limits aswell and an abelian presheaf is a sheaf if and only if the underlying presheaf of sets is a sheaf as discussed above. Therefore we automatically get a well behaved theory of abelian sheaves. It is clear that $PSh(\mathbf{C}, Ab)$ also is an abelian category. This also implies that $Sh(\mathbf{C}, Ab)$ is an abelian category through exploitation of the sheafification functor. We will not give details of all the constructions. The only subtle point to remember is that after doing the obvious construction in $PSh(\mathbf{C}, Ab)$ sectionwise, one should sheafify whenever colimits are involved to land back again in $Sh(\mathbf{C}, Ab)$.

Definition 1.4.13. Consider a pair $(\mathbf{C}, \mathcal{O})$ where $\mathbf{C}$ is a site and $\mathcal{O}$ is a sheaf of rings on $\mathbf{C}$. We will call $(Sh(\mathbf{C}), \mathcal{O})$ a ringed topos. There is a notion of morphism between ringed topoi. A morphism of ringed topoi is a pair

$$(f, f^{\#}) : (Sh(\mathbf{C}), \mathcal{O}) \to (Sh(\mathbf{C}'), \mathcal{O}')$$

where $f : \mathbf{C} \to \mathbf{C}'$ is a morphism of sites together and a map of sheaves of rings $f^{\#} : f^{-1}\mathcal{O}' \to \mathcal{O}$ or by adjunction $\mathcal{O}' \to f_*\mathcal{O}$

Later, for convenience we will drop Ab from the notation $Sh(\mathbf{C}, Ab)$ whenever it is clear from context whether we are talking about sheaves valued in abelian groups.

Definition 1.4.14. Given a topos $Sh(\mathbf{C})$ and a sheaf of rings $\mathcal{O}$) on it; a sheaf of $\mathcal{O}$-modules is a presheaf of $\mathcal{O}$-modules (i.e a presheaf F such that there is a map $\mathcal{O} \times F \to F$ satisfying the usual axioms) whose underlying presheaf of abelian groups is a sheaf. Morphism of sheaves of modules is just a morphism as presheaves of modules. The category of sheaves of $\mathcal{O}$-modules will be denoted $Mod(\mathcal{O})$.

There is a notion of tensor product, defined on presheaves using sections and extended to sheaves by sheafification. We will use it without giving details. We refer the interested reader to [Stacks, Tag 03A4].

Suppose we have a morphism of topoi $f : Sh(\mathbf{C}) \to Sh(\mathbf{D})$, Let $\mathcal{O}$ be a sheaf of rings on $\mathbf{C}$ and $\mathcal{F}$ a sheaf of $\mathcal{O}$-modules. Since f_* commutes with limits, $f_*\mathcal{O} \times f_*\mathcal{F} \to f_*\mathcal{F}$ makes $f_*\mathcal{F}$ into a sheaf of $f_*\mathcal{O}$ modules. Similarly for f^{-1}.

Proposition 1.4.15. *Then we have the following natural isomorphism*

$$\mathrm{Hom}_{Mod(f^{-1}\mathcal{O})}(f^{-1}\mathcal{G}, \mathcal{F}) = Mor_{Mod(\mathcal{O})}(\mathcal{G}, f_*\mathcal{F})$$

Where $f_*\mathcal{F}$ thought of as an $\mathcal{O}$-module by restriction from $\mathcal{O} \to f_*f^{-1}\mathcal{O}$

Proof. We have
$$Mor_{Sh(\mathbf{C},Ab)}(f^{-1}\mathcal{G},\mathcal{F}) = Mor_{Sh(\mathbf{C},Ab)}(\mathcal{G},f_*\mathcal{F}).$$
Then, it will suffice to show under this natural isomorphism, that $f^{-1}\mathcal{O}$-linear maps in left hand side correspond to $\mathcal{O}$-linear maps in the right hand side. Suppose $\alpha : f^{-1}\mathcal{G} \to \mathcal{F}$ and $\beta : \mathcal{G} \to f_*\mathcal{F}$ are morphisms of abelian sheaves which correspond to each other under the above natural isomorphism. We will be done if we can show α is $f^{-1}\mathcal{O}$-linear if and only if β is $\mathcal{O}$-linear. Suppose, $\alpha : f^{-1}G \to \mathcal{F}$ is $f^{-1}\mathcal{O}$-linear, meaning that the below diagram commutes.

$$\begin{tikzcd}
f^{-1}\mathcal{O} \times f^{-1}\mathcal{G} \arrow[r, "1\times\alpha"] \arrow[d] & f^{-1}\mathcal{O} \times \mathcal{F} \arrow[d] \\
f^{-1}\mathcal{G} \arrow[r, "\alpha"] & \mathcal{F}
\end{tikzcd}$$

Then by functoriality and the fact f_* commutes with limits, $f_*\alpha$ is $f_*f^{-1}\mathcal{O}$-linear. Hence it is also $\mathcal{O}$-linear.

$$\begin{tikzcd}
G \arrow[rr, "\beta"] \arrow[dr, "\eta"'] & & f_*F \\
& f_*f^{-1}G \arrow[ur, "f_*\alpha"'] &
\end{tikzcd}$$

We know $f_*\alpha$ is $\mathcal{O}$-linear. Therefore if η is also $\mathcal{O}$-linear β will be $\mathcal{O}$-linear. Finally, η is $\mathcal{O}$-linear since f_* and f^{-1} both commute with products and it is a transformation of functors.

Similarly, starting by supposing β is $\mathcal{O}$-linear, we can repeat the argument for the diagrams below

$$\mathcal{G} \xrightarrow{\beta} f_*\mathcal{F}$$

$$f^{-1}\mathcal{G} \xrightarrow{f^{-1}\beta} f^{-1}f_*\mathcal{F}$$

$$\begin{tikzcd}
f^{-1}\mathcal{G} \arrow[rr, "\alpha"] \arrow[dr, "f^{-1}\beta"] & & \mathcal{F} \\
& f^{-1}f_*\mathcal{F} \arrow[ur, "\epsilon"] &
\end{tikzcd}$$

so the question reduces to showing it for the counit of adjunction ϵ. It $f^{-1}\mathcal{O}$-linear with exactly the same argument. ∎

Remark 1.4.16. Given a morphism of ringed topoi $(f, f^\#) : (Sh(\mathbf{C}), \mathcal{O}_C) \to (Sh(\mathbf{D}), \mathcal{O}_D)$, we can define pullback and pushforward in the usual way and they are adjoint. To define the pushforward, restrict through the morphism $\mathcal{O}_D \to f_*\mathcal{O}_C$. To define the pullback take the tensor product.

$$\begin{aligned} f^* : &Mod(\mathcal{O}_C) \to Mod(\mathcal{O}_D) \\ F &\mapsto \mathcal{O}_C \otimes_{f^{-1}\mathcal{O}_D} f^{-1}F \end{aligned}$$

These are adjoint as we expect. Indeed:

$$\begin{aligned} \mathrm{Hom}_{Mod(\mathcal{O}_C)}(f^*G, F) &\xrightarrow{\sim} \mathrm{Hom}_{Mod(\mathcal{O}_C)}(\mathcal{O}_C \otimes_{f^{-1}O_D} f^{-1}G, F) \\ &\xrightarrow{\sim} \mathrm{Hom}_{Mod(f^{-1}\mathcal{O}_D)}(f^{-1}G, F_{f^{-1}\mathcal{O}_D}) \\ &\xrightarrow{\sim} \mathrm{Hom}_{Mod(\mathcal{O}_D)}(G, f_*F) \end{aligned}$$

Now we will use the lower shriek to obtain a generating set for our abelian category $Mod(\mathcal{O})$. Given a site $\mathbf{C}$ and an object U of $\mathbf{C}$, the over category $\mathbf{C}/U$ inherits in the obvious way the structure of a site. There is a functor $j_U : C/U \to C$ which is continuous and cocontinuous. By the previous discussion, we get three functors $j_{U!} \dashv j^{-1} \dashv j_*$. In such context j^{-1} is usually denoted by $\bullet|_U$ and called restriction for obvious reasons. Moreover, if $(Sh(\mathbf{C}), \mathcal{O})$ is a ringed site, by definining $\mathcal{O}_U := j^{-1}\mathcal{O}$ and using the obvious module structures this gives a functor $Mod(\mathcal{O}_U) \to Mod(\mathcal{O})$. This is called extension by zero because any object not admitting a morphism to U gets assigned the 0 by the sheaves that are in the image of this functor. Straightforward calculation shows this functor to be exact. For details check [Stacks, Tag 03DH].

The significance of these functors is that they form a generating set for the category. One consequence of this is that any object $\mathcal{F}$ in $Mod(\mathcal{O})$ admits a surjection from a direct sum of these. To see why just notice that we have morphisms

$$\bigoplus_{s \in F(U)} \mathcal{O}_U \to F|_U$$

using the adjunction and running through all the objects we get a morphism

$$\bigoplus_{(U,s)} j_{U!}\mathcal{O}_U \to F$$

It is a surjection by construction.

Using this we can define a lower shriek for modules.

Proposition 1.4.17. *Let $u : \mathbf{C} \to \mathbf{D}$ be a continuous and cocontinuous functor between sites and $f : Sh(\mathbf{C}) \to Sh(\mathbf{D})$ the associated morphism of topoi. Let $\mathcal{O}_D$ be a sheaf of rings in $Sh(\mathbf{D})$ and set $\mathcal{O}_C = g^{-1}\mathcal{O}_D$. Then f induces a morphism of ringed topoi with $f^* = f^{-1}$. There exist a functor*

$$f_! : Mod(\mathcal{O}_C) \to Mod(\mathcal{O}_D)$$

with $f_! \dashv f^*$

Proof. If U is an object of $\mathbf{C}$ we have

$$\begin{aligned} \mathrm{Hom}_{\mathcal{O}_C}(j_{U!}\mathcal{O}_U, f^{-1}\mathcal{G}) &\xrightarrow{\sim} f^{-1}\mathcal{G}(U) \\ &\xrightarrow{\sim} \mathcal{G}(u(U)) \\ &\xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}_D}(j_{u(U)!}\mathcal{O}_{u(U)}, G) \end{aligned}$$

Let us denote sheaves of the form $\bigoplus_I j_{U!}\mathcal{O}_U$ by $\mathcal{P}_i$ for some arbitrary index set I. For sheaves of this type, define $f_!\mathcal{P}_i := f_! \bigoplus j_{U!}\mathcal{O}_U := \bigoplus j_{u(U)!}O_{u(U)}$

Given an $\mathcal{F}$ in $Mod(\mathcal{O}_C)$ we can find an exact sequence

$$\mathcal{P}_2 \to \mathcal{P}_1 \to \mathcal{F} \to 0$$

Define $f_!(\mathcal{F}) :=$ Coker $(f_!\mathcal{P}_2 \to f_!\mathcal{P}_1)$ Taking Hom of the presentations of $\mathcal{F}$ and $f_!\mathcal{F}$, we get an exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Hom}_{\mathcal{O}_C}(\mathcal{F}, f^{-1}\mathcal{G}) \arrow[r] & \mathrm{Hom}_{\mathcal{O}_C}(\mathcal{P}_1, f^{-1}\mathcal{G}) \arrow[r] \arrow[d, "\wr"'] & \mathrm{Hom}_{\mathcal{O}_C}(\mathcal{P}_2, f^{-1}\mathcal{G}) \arrow[d, "\wr"'] \\ 0 \arrow[r] & \mathrm{Hom}_{\mathcal{O}_D}(f_!\mathcal{F}, \mathcal{G}) \arrow[r] & \mathrm{Hom}_{\mathcal{O}_D}(f_!\mathcal{P}_1, \mathcal{G}) \arrow[r] & \mathrm{Hom}_{\mathcal{O}_D}(f_!\mathcal{P}_2, \mathcal{G}) \end{tikzcd}$$

Showing that the adjunction holds ∎

Remark 1.4.18. Beware that due to the nature of this construction, the diagram below does not generally commute.

$$\begin{tikzcd} Mod(O_C) \arrow[r, "f_!"] \arrow[d, "forgetful"] & Mod(O_D) \arrow[d, "forgetful"] \\ Ab(\mathcal{C}) \arrow[r, "f_!^{Ab}"] & Ab(\mathcal{D}) \end{tikzcd}$$

Where $g_!^{Ab}$ is the same construction done for $Mod(\underline{\mathbb{Z}})$ the modules of the constant sheaf of rings on $\mathbb{Z}$.

There is a transformation of functors

$$f_!^{Ab} \circ forgetful \to forgetful \circ f_!$$

and it commutes if and only if

$$f_!^{Ab} j_{U!}\mathcal{O}_U \to f_! j_{U!}\mathcal{O}_U$$

is an isomorphism for all objects U. Since $f_! j_{U!}\mathcal{O}_U := j_{u(U)!}\mathcal{O}_{u(U)}$ the diagram commutes if and only if

$$f_!^{Ab} j_{U!}\mathcal{O}_U \xrightarrow{\sim} j_{u(U)!}\mathcal{O}_{u(U)}.$$

Moreover, we can derive lower shriek. A priori, it is not clear that we have projectives so we need to provide an argument as to why we can find a $\bullet_!$ adapted class of objects. This is essentially a computational exercise to show the most obvious candidate, modules of the form $\bigoplus j_{U!}\mathcal{O}_U$, work. For our purposes, these objects will actually be projective as we will work with a chaotic topology where taking sections is an exact functor. With this given, it should be easy to see an analogous statement for the map $Lf_!^{Ab} \circ forgetful \to forgetful \circ Lf_!$ to be an isomorphism. It is the case if and only if

$$Lf_!^{Ab} j_{U!}\mathcal{O}_U \xrightarrow{\sim} j_{u(U)!}\mathcal{O}_{u(U)}$$

for all U. We will content ourselves with knowing the existence and this fact. We refer the interested reader to [Stacks, Tag 07AB] for the details.

1.5 Homology on a category

Throughout this subsection, $\mathbf{C}$ is a site endowed with the chaotic topology. Therefore sheaves and presheaves on $\mathbf{C}$ agree. Beware that in this section, we will be using homological notation.

There is a canonical functor $p : \mathbf{C} \to *$ where $*$ denotes the category with a single object and single morphism equipped also with the chaotic topology. Then this functor is continuous and cocontinuous thus induces a map $\pi : Sh(\mathbf{C}) \to Sh(*)$. Let B be a ring and denote by $\underline{B} := \pi^{-1}B$ the constant (pre)sheaf on B. Then this induces a morphism of ringed topoi

$$\pi : (Sh(\mathbf{C}), \underline{B}) \to (Sh(*), B)$$

Lemma 1.5.1. *In this case,* $\pi_!(F) = \operatorname{colim}_{\mathbf{C}^{op}} F$.

Proof. A map $\operatorname{Hom}_{Mod(\underline{B})}(F, \pi^{-1}G)$ is precisely the data making G a cocone for F as a functor from C^{op} to $Mod(B)$. Therefore it functorialy corresponds to a unique map $\operatorname{Hom}_{Mod(B)}(\operatorname{colim}_{C^{op}} F, G)$ which then proves our result by uniqueness of adjoint functors. $\square$

This implies $L_n\pi_!$ is the $n-th$ left derived functor of taking colimits. Note that, since taking colimits commutes with the forgetful functor to abelian groups, in this case $\pi_! = \pi_!^{Ab}$. Moreover, denoting $\underline{B}$ by $\mathcal{O}$, observe that $L\pi_!^{Ab}(j_{U!}\mathcal{O}_U) = \pi_!^{Ab} j_{U!}\mathcal{O}_U = \pi_! j_{U!}\mathcal{O}_U = j_{\pi(U)!}\mathcal{O}_{\pi(U)}$. Then it follows by Remark 1.4.18 that in this case $L\pi_!^{Ab}$ agrees with $L\pi_!$ after forgetting the module structure. In this situation, one can use simplicial methods to compute it.

Proposition 1.5.2. *Let $S^\bullet$ be a cosimplicial object in $\mathbf{C}$. (In otherwords a covariant functor $S : \Delta \to \mathbf{C}$). If for every object U in $\mathbf{C}$ the simplicial set $\mathrm{Hom}_{\mathbf{C}}(S^\bullet, U)$ is homotopy equivalent to the point (the constant simplicial set on a singleton) then for all F in $(Sh(\mathbf{C}, \underline{B})$ the following holds:*

$$L\pi_!(F) = F(S^\bullet)$$

Here, the right hand side stands for the complex associated to the simplicial object through the Dold-Kan equivalence.

Proof. It will suffice to do it for $L\pi_!^{Ab}$. For any object U of C, the sheaf $j_{U!}\mathbf{Z}_U$ (where $\mathbf{Z}$ is the constant sheaf with values $\mathbb{Z}$) is projective in $Ab(\mathbf{C})$ because $\mathrm{Hom}(j_{U!}\mathbf{Z}_U, F) = F(U)$ and taking sections is exact (the topology is chaotic). We know that every abelian sheaf is a quotient of a direct sum of modules of the form $j_{U!}\mathbf{Z}_U$. Lets label these by P^i. Then we can take a resolution of F in the form of

$$\cdots \to P_{-1} \to P_0 \to F \to 0$$

and compute $L\pi_!(F) = \pi_!(P_\bullet)$. Now, we have a double complex $A_{\bullet,\bullet} := P_\bullet(S^\bullet)$ where columns are understood to be the complexes associated to the simplicial object. Then the map $P^0 \to F$ induces a map of complexes $A_{0,\bullet} \to F(S_\bullet)$. Notice that, as the rows are exact, the second spectral sequence of the double complex shows $Tot(A_{\bullet,\bullet})$ is quasi-isomorphic to the complex associated to $F(S_\bullet)$. Moreover, since $\pi_!$ is computed by taking colimits, we have canonical maps $A_{\bullet,m} \to \pi_!(P_m)$. These induce the following map

$$Tot(A_{\bullet,\bullet}) \to \pi_!(P_\bullet) = L\pi_!(F).$$

We will prove the columns are exact, showing the total complex is quasi-isomorphic to $L\pi_!(F)$ by use of the first spectral sequence of the double complex. The columns look like the following

$$\cdots \to P_m(S_1) \to P_m(S_0) \to \pi_! P_m \to 0$$

Since P_i were direct sums of sheaves of form $j_{U!}\mathbf{Z}_U$, we can reduce to that case. Now by the explicit construction of $j_{U!}$ and the fact there is no sheafification involved implies that the complex associated to $j_{U!}\mathbf{Z}_U(S^\bullet)$ is the complex of abelian groups associated to the free $\mathbf{Z}$ module on $\mathrm{Hom}_C(S^\bullet, U)$ which was assumed to be homotopic to the point. Therefore it is the complex with $\mathbf{Z}$ in each degree with diferentials alternating between identity and 0 (i.e the singular homology complex of a point). As $\pi_! j_U \mathbf{Z}_U = \mathbf{Z}$, it fixes the defect in degree 0 and the columns are exact as expected. □

Remark 1.5.3. This proposition provides another interpretation of $L\pi_!$. Consider $C = \Delta$ and $S^\bullet = Id_C$. If B is a ring, then $Mod(\underline{B})$ is the category of simplicial B modules and $L\pi_!$ takes a simplicial module to its associated complex under the Dold-Kan correspondance.

Here we prove a slightly technical result which will let us prove certain simplicial objects are resolutions.

Proposition 1.5.4. *Let $S^\bullet$ be a cosimplicial object like before. Let $\mathcal{O} \to \mathcal{O}'$ be a map of sheaves of rings on $\mathbf{C}$. If $L\pi_!\mathcal{O} \to L\pi_!\mathcal{O}'$ is an isomorphism. then for K in $D(\mathcal{O})$ we have*

$$L\pi_!(K) = L\pi_!(K \otimes^{\mathbf{L}}_{\mathcal{O}} \mathcal{O}')$$

in $D(Ab)$

Proof. Since $j_{U!}\mathcal{O}_U$ are our projective objects in $D(\mathcal{O})$, it will suffice to prove it for them. We know that

$$HS_\bullet := \mathrm{Hom}_{\mathbf{C}}(S^\bullet, U)$$

is homotopy equivalent to the point. Let $P_\bullet := \mathcal{O}(S^\bullet)$ and $P'_\bullet := \mathcal{O}'(S^\bullet)$. Notice that by the previous proposition and the explicit description of $j_{U!}\mathcal{O}_U$ the complexes associated to the simplicial abelian groups

$$X_\bullet : n \mapsto \bigoplus_{s \in HS_n} P_n = HS_\bullet \otimes \mathcal{O}(S^\bullet)$$

$$X'_\bullet : n \mapsto \bigoplus_{s \in HS_n} P'_n = HS_\bullet \otimes \mathcal{O}'(S^\bullet)$$

compute $L\pi_!(j_{U!}\mathcal{O}_U)$ and $L\pi_!(j_{U!}\mathcal{O}'_U)$ respectively. But note that since $HS_\bullet$ is homotopy equivalent to a point, the complex associated to $X_\bullet$ is homotopic to the one associated to $P_\bullet$. Likewise with with $X'_\bullet$ and $P'_\bullet$. Now we are almost done. Notice that since $j_{U!}\mathbf{O}_U$ is a flat $\mathcal{O}$ module,

$$j_{U!}\mathcal{O}_U \otimes^{\mathbf{L}}_{\mathcal{O}} \mathcal{O}'_U = j_{U!}\mathcal{O}_U \otimes_{\mathcal{O}} \mathcal{O}'_U = j_{U!}\mathcal{O}'_U$$

By assumption complexes associated to $P_\bullet$ and $P'_\bullet$ are quasi-isomorphic since they compute $L\pi_!\mathcal{O}$ and $L\pi_!\mathcal{O}'$ respectivel respectively. Then we see the associated complexes of $X_\bullet$ and $X'_\bullet$ are quasi-isomorphic which was what we wanted to show. □

2 Cotangent complex and smooth morphisms

In this section we begin by recalling basic notions of differential calculus on schemes and then talk a bit about deformation theory in the smooth setting. Beware, in the reference we have given for the Dold-Kan correspondance, homological conventions are used (as is standard). However as we will be using cohomological notation, for the sake of consistency, we will replace n with $-n$ in the complex associated to a simplicial abelian group.

2.1 Kähler differentials

Definition 2.1.1. A closed immersion $i : T_0 \to T$ is called a *thickening of order 1* if it is defined by an ideal of square zero.

Given a thickening i as in the definition, its ideal $\mathcal{J}$, has a structure of a $\mathcal{O}_{T_0} = \mathcal{O}_T/\mathcal{J}$ module structure since $\mathcal{J}$ is annihilated by itself. Moreover, given $j : X \to Z$ an immersion with ideal $\mathcal{I}$, it factors uniquely through a thickening

$$X \overset{j_1}{\to} Z_1 \overset{h_1}{\to} Z$$

where Z_1 has the same underlying space as X but is defined by the ideal $\mathcal{I}^2$. Then h_1 is an immersion and j_1 is a thickening with ideal $\mathcal{I}/\mathcal{I}^2$. This ideal is a quasi-coherent $\mathcal{O}_X$-module and we will denote it by $\mathcal{N}_{X/Z}$.

Let $f : X \to Y$ be a morphism of schemes and let $\Delta : X \to Z := X \times_Y X$ be the diagonal which is an immersion. The conormal sheaf of Δ is called the sheaf of Kähler 1-differentials of X over Y. It is denoted by $\Omega^1_{X/Y}$. Factor Δ as $X \to Z_1 \to X$ just as before. Then the projections from Z to X induce two morphisms $Z_1 \to Z$. The structure sheaf of Z_1 is called the sheaf of principal parts of order 1 of X over Y. It will be denoted by $\mathcal{P}^1_{X/Y}$. Then by the construction there is an exact sequence

$$0 \to \Omega^1_{X/Y} \to \mathcal{P}^1_{X/Y} \to \mathcal{O}_X \to 0$$

which is split by homomorphisms $j_1, j_2 : \mathcal{O}_X \to \mathcal{P}^1_{X/Y}$ induced from the projections. The difference $j_2 - j_1$ is a homomorphism $\mathcal{O}_X \to \Omega^1_{X/Y}$ called the differential. It will be denoted by d or $d_{X/Y}$.

Given a $\mathcal{O}_X$-module M, a Y-derivation of X valued in M is a morphism of sheaves of $f^{-1}\mathcal{O}_Y$ modules $D : \mathcal{O}_X \to M$ satisfying

$$D(ab) = aD(b) + bD(a)$$

for sections and $D(f^{-1}\mathcal{O}_Y) = 0$. Denote by $\mathrm{Der}_Y(\mathcal{O}_X, M)$ the set of Y-derivations valued in M. There is an obvious abelian group structure on it obtained by addition. Moreover regarded as a functor in M, it is representable. The map

$$\mathrm{Hom}(\Omega^1_{X/Y}, M) \to \mathrm{Der}_Y(\mathcal{O}_X, M), \quad u \mapsto u \circ d_{X/Y}$$

gives a natural isomorphism. For now we don't prove this, it will be obvious soon when we give another description of $\Omega^1_{X/Y}$.

Suppose X and Y are affine, say $\operatorname{Spec} B$, $\operatorname{Spec} A$ respectively. Denote by I the kernel $\operatorname{Ker}(B \otimes B \to B)$. Then $\Omega^1_{X/Y} = I/I^2$ and $d_{X/Y}(b) = b \otimes 1 - 1 \otimes b$. Alternatively, one can construct $\Omega^1_{X/Y}$ as the free B-module on the symbols db modulo the relations $d(b) + d(b') = d(b + b')$, $d(bc) = bdc + cdb$, if $a \in A$ $da = 0$. This description makes the previous discussion about representability obvious. By the end of the section the equivalence of these characterizations will be clear. We will continue the discussion in the affine case. How to globalize should be straightforward.

Given a commutative square of ring maps

$$\begin{tikzcd} B \arrow[r, "\varphi"] & B' \\ A \arrow[u] \arrow[r, "\psi"] & A' \arrow[u] \end{tikzcd},$$

there are natural maps making the below diagram commutative

$$\begin{tikzcd} \Omega_{B/A} \arrow[r] & \Omega_{B'/A'} \\ B \arrow[u, "d_{B/A}"] \arrow[r, "\varphi"] & B' \arrow[u, "d_{B'/A'}"'] \end{tikzcd}$$

where the upper horizontal map is given by $d(b) \mapsto d(\varphi(b))$.

Attached to certain ring maps, we have two important short exact sequences

Proposition 2.1.2. *Let $A \to B \to C$ be ring maps. Then we have a exact sequence of C-modules*

$$C \otimes_B \Omega_{B/A} \to \Omega_{C/A} \to \Omega_{C/B} \to 0$$

Proof. It is clear that the obvious map $\Omega_{C/A} \to \Omega_{C/B}$ is surjective. The map $C \otimes_B \Omega_{B/A} \to \Omega_{C/A}$ is given by $c \otimes db \mapsto cdb$. Now if dc is in the kernel, then by the presentation of $\Omega_{C/B}$, dc is a finite linear combination symbols of the form. db, $d(c + c') - d(c) - d(c')$, $d(c_i c_j) - c_i d(c_j) - c_j d(c_i)$. Clearly, symbols of the latter two form are also 0 in $\Omega_{C/A}$, therefore the kernel is generated by the image of $C \otimes_B \Omega_{B/A}$ which gives us the short exact sequence. □

Proposition 2.1.3. *Let $A \to B \to C$ be ring maps with $B \to C$ surjective with kernel I. Then we have a exact sequence of C-modules*

$$I/I^2 \to C \otimes_B \Omega_{B/A} \to \Omega_{C/A} \to 0\ ,$$

where the leftmost map is given by $f \mapsto 1 \otimes df$.

Proof. Notice this is just an extension of the short exact sequence above, as in this case $\Omega^1_{C/B} = 0$. Only tricky thing to show is the that the kernel is the image of I/I^2. An element in the kernel is a finite linear combination of symbols of the form da, $d(c+c') - d(c) - d(c')$, $d(c_i c_j) - c_i d(c_j) - c_j d(c_i)$. Also note that, fixing a random section of $B \to C$ as a map of sets, we can write each $b \in B$ as $c + i$ where $\operatorname{im}(c) = b$ and $i \in I$. Therefore, the kernel can be expressed precisely as finite linear combinations of $d(c + i + c' + i') = d(c + i) - d(c' + i')$, $d((c_j + i_k)(c_i + i_j)) - (c_i + i_k)d(c_j + i_j) - (c_j + i_j)d(c_i + i_j)$. From this it is straightforward to see the image of I/I^2 is the kernel. □

Now, we look at what happens under passing to localizations, which will be crucial when we talk about smoothness at points.

Proposition 2.1.4. *Let $\varphi : A \to B$ be a ring map. If $S \subset A$ is a multiplicative subset, mapping to invertible elements of B then $\Omega_{B/A} = \Omega_{B/S^{-1}A}$.*

If $T \subset B$ is a multiplicative subset of B then $T^{-1}\Omega_{B/A} = \Omega_{T^{-1}B/A}$

Proof. Any $S^{-1}A$ derivation is naturally an A derivation. Notice that given an A derivation $d : B \to M$ $d(1) = d(\varphi(s)\varphi(s)^{-1}) = \varphi(s)d\varphi(s)^{-1} + \varphi(s)^{-1}d\varphi(s) = \varphi(s)d\varphi(s)^{-1} = 0$ implying $d\varphi(s)^{-1} = 0$ as $\varphi(s)$ is invertible. Thus any A derivation is naturally also an $S^{-1}A$ derivation proving the first assertion.

There is a map $T^{-1}\Omega_{B/A} \to \Omega_{T^{-1}B/A}$ given by $\frac{b}{t}db \to \frac{b}{t}db$. To find the inverse, notice that to any derivation $m : T^{-1}\Omega_{B/A} \to M$ we can associate a $T^{-1}B$ derivation by $d(\frac{b}{t}) := \frac{m(b)}{t} - \frac{bm(t)}{t^2}$. $\square$

The main takeaway from this is, given a map of schemes $X \to Y$ and points $\mathfrak{p} \mapsto \mathfrak{q}$, $\Omega^1_{X_\mathfrak{p}/Y_\mathfrak{q}} = (\Omega^1_{X/Y})_\mathfrak{p}$.

The next two propositions make it possible to compute and work with differentials.

Proposition 2.1.5. *Take two A algebras B and C. Then $\Omega^1_{B\otimes_A C/B} \cong \Omega^1_{C/A} \otimes_C (B \otimes_A C)$.*

Proof. For simplicity let us denote the base change of C by B as C'. There is a B derivation $C' \to \Omega^1_{C/A} \otimes_A B$ given by $b \otimes c \mapsto d(c) \otimes b$. Given any B derivation $m : C' \to M$, we get a unique induced map $\Omega^1_{C/A} \otimes_A B \to M$ given by $dc \otimes b \mapsto m(c \otimes b)$. Therefore it satisfies the universal property of $\Omega^1_{C'/B}$ implying they are isomorphic. $\square$

The key takeaway is, module of differentials is compatible with base change.

Proposition 2.1.6. *If $B = A[x_1, \cdots, x_n]$ then $\Omega^1_{B/A} \cong \bigoplus_{i=1}^n B \cdot dx_i$ as a B-module.*

Proof. By induction, we can reduce it to the case $B = A[x]$. Here we have the map $\Omega^1_{B/A} \to A \cdot dx$ given by $df(x) \mapsto f'(x)dx$. This is an isomorphism because if $f'(x) = 0$ then $f \in A$ which means $df(x) = 0$. $\square$

This combined with the short exact sequences we have, implies finite type ring maps have finite type modules for their differentials. Similarly, finitely presented ring maps have finitely presented modules for their differentials.

2.2 The cotangent complex of a ring map

Let A be a ring and Alg_A denote the category of A-algebras. We have an adjunction $U \dashv V$ where $V : Alg_A \to Sets$ is the forgetful functor and U is the free A-algebra functor. We have seen in example 1.3.4 that this gives us a simplicial object $X_\bullet \in Func(Alg_A, Alg_A)$.

Definition 2.2.1. Given a ring map $A \to B$, denote by $P_\bullet$ the simplicial A-algebra $X_\bullet(B)$. This will be called the standard resolution. There is a canonical augmentation (map to constant simplicial object with value B) $\epsilon : P_\bullet \to B$. The *cotangent complex* $L_{B/A}$ is defined to be the complex associated to the simplicial B-module

$$\Omega_{P_\bullet/A} \otimes_{P_\bullet,\epsilon} B$$

Let $A \to B$ be a ring map. Define a category whose objects are A-algebra maps $\alpha : P \to B$ where P is a free algebra over A and morphisms are A-algebra morphisms making the diagram below commutative.

$$\begin{tikzcd} P \arrow[r] \arrow[d] & P' \arrow[dl] \\ B & \end{tikzcd}$$

Let $C := C_{B/A}$ denote the opposite of this category. Make C a site with the chaotic topology. There are two sheaves of rings on C. One is the constant presheaf $\underline{B}$ and the other one given by $\mathcal{O}(\alpha : P \to B) := P$, with obvious definitions on morphisms. Then we get the following diagram of morphisms of ringed topoi

$$\begin{tikzcd} (Sh(C), \underline{B}) \arrow[r, "i"] \arrow[d, "\pi"] & (Sh(C), \mathcal{O}) \\ (Sh(*), B) & \end{tikzcd}$$

i is the identity on the underlying topoi and $i^\# : \mathcal{O} \to \underline{B}$ is the obvious map.

Lemma 2.2.2. *Let $P_\bullet$ be a simplicial A-algebra endowed with an augmentation $\epsilon : P_\bullet \to B$ with each P_n a polynomial algebra over A and ϵ is a homotopy equivalence on the underlying simplicial sets. Then for F in $Mod(\underline{B})$ the following holds:*

$$L\pi_!(F) = F(P_\bullet, \epsilon)$$

Proof. We will use Proposition 1.5.2. Given an object (Q, β) of C we need to show

$$\mathrm{Hom}_C((Q, \beta), (P_\bullet, \epsilon))$$

is homotopy equivalent to a singleton. $Q = A[E]$ for some set by construction. Define S to be the category analogous to C but instead of A-algebra maps to B, using set maps to B. Then by free, forgetful adjunction

$$\mathrm{Hom}_C((Q, \beta), (P_\bullet, \epsilon)) \cong \mathrm{Hom}_S((E \to B), (V(P_\bullet \to B))$$

But by assumption $V(P_\bullet \to B) \to V(B \to B)$ is a homotopy equivalence (maps are the obvious ones coming from the equivalence of $\epsilon : P_\bullet \to B$). Therefore $\mathrm{Hom}_C((Q, \beta), (P_\bullet, \epsilon))$ is homotopy equivalent to

$$\mathrm{Hom}_S((E \to B), V(B \to B)) = \{*\}$$

and we are done. $\square$

By example 1.3.9 the standard resolution is of this form. This suggests, really the correct definition of the cotangent complex should be taken using the derived lower shriek as in the below lemma. However the standard resolution is a particularly nice resolution to work with.

Lemma 2.2.3. *With notation as above, there are canonical isomorphisms*

$$L_{B/A} = L\pi_!(Li^*\Omega_{\mathcal{O}/A}) = L\pi_!(i^*\Omega_{\mathcal{O}/A}) = L\pi_!(\Omega_{\mathcal{O}/A} \otimes_{\mathcal{O}} \underline{B})$$

in $D(B)$.

Proof. $\Omega_{\mathcal{O}/A}$ is a flat $\mathcal{O}$-module since $\Omega_{P/A}$ is a free a P-module for each $\alpha : P \to B$. Therefore we don't have to derive i^*. Therefore $Li^*\Omega_{\mathcal{O}/A} = i^*\Omega_{\mathcal{O}/A}$ is the sheaf of $\underline{B}$-modules which associates to each $\alpha : P \to B$ the B-module $\Omega_{P/A} \otimes_{P,\alpha} B$. By the above lemma the right hand side can be computed by the value of this sheaf on the standard resolution, which by definition is the left hand side. $\square$

Lemma 2.2.4. $H^0(L_{B/A}) = \Omega_{B/A}$.

Proof. There is an obvious map from $\Omega_{\mathcal{O}/A} \otimes \underline{B}$ to the constant presheaf with value $\Omega_{B/A)}$. This induces a map

$$H^0(L_{B/A}) = H^0(L\pi_!(\Omega_{\mathcal{O}/A} \otimes \underline{B})) = \pi_!(L\pi_!(\Omega_{\mathcal{O}/A} \otimes \underline{B})) \to \Omega_{B/A}$$

Picking an object $P \to B$ with $P \to B$ we can see that the map is surjective since $\Omega_{P/A} \otimes B \to \Omega_{B/A}$ is surjective. To show it is injective is trickier. For some $P \to B$ an object of C suppose $\eta \in \Omega_{P/A} \otimes B$ maps to zero. Factoring it as $P \to P' \to B$ with $P' \to B$ surjective, we see there is no loss of generality on assuming $P \to B$ is surjective. If $B = P/I$ then η comes from some element $f \in I/I^2$. Consider maps $0, f : P[x] \to P$ which take x to 0 and f respectively. Then both η and 0 are images of $dx \otimes 1$ in $\Omega_{P'/A} \otimes_{P'} B$. Therefore η is 0 in the colimit showing the map is injective. $\square$

If B happens to be a free algebra over A then by Lemma 2.2.2, the constant simplicial object B can be used to compute $L\pi_!$. As a corollary in this case $L_{B/A}$ is quasi-isomorphic to $\Omega_{B/A}[0]$.

From the functoriality of the standard resolution, given a square of rings

$$\begin{tikzcd} B \arrow[r] & B' \\ A \arrow[u] \arrow[r] & A' \arrow[u] \end{tikzcd}$$

we get a map $L_{B/A} \to L_{B'/A'}$.

Moreover, the cotangent complex is compatible with base change.

Lemma 2.2.5. *If the above diagram induces $B' \cong B \otimes^{\mathbf{L}}_A A'$. Then we have an isomorphism*

$$L_{B/A} \otimes^{\mathbf{L}}_B B' \to L_{B'/A'}$$

Proof. Let $P_\bullet \to B$ be the standard resolution of B over A. Then by definition $L_{B/A}$ is the complex associated to $\Omega_{P_\bullet/A} \otimes_{P_\bullet} B$. Notice that, components are each free and consequently flat B modules. Therefore to calculate $L_{B/A} \otimes^{\mathbf{L}}_B B'$, we can simply tensor each component by B'. Denote by $P'_\bullet := P_\bullet \otimes_A A'$. Then by the compatibility of differentials by base change we see that $\Omega_{P'_\bullet/A'} = \Omega_{P_\bullet/A} \otimes_{P_\bullet} P'_\bullet$. Thus we have

$$L_{B/A} \otimes^{\mathbf{L}}_B B' = \Omega_{P_\bullet/A} \otimes_{P_\bullet} B \otimes_B B' = \Omega_{P'_\bullet/A'} \otimes_{P'_\bullet} B'$$

Notice that P'_n is a free A' algebra and by hypothesis the complex associated to $P_\bullet \otimes_A A'$ is quasi-isomorphic to the complex associated to the constant simplicial module B'. By, [Stacks, Tag 08NS] and [Stacks, Tag 08P1] combined, this implies they are even homotopy equivalent. Therefore the complex associated to $\Omega_{P'_\bullet/A'} \otimes_{P'_\bullet} B'$ computes $L_{B'/A'}$ and we are done. □

Corollary 2.2.6. *If $B' = B$ then $L_{B/A} \cong L_{B/A'}$.*

2.3 Fundamental triangle

Our goal in this subsection is to prove the following

Theorem 2.3.1 (The fundamental triangle)**.** *Let $A \to B \to C$ be ring maps. Then there is a distinguished triangle*

$$L_{B/A} \otimes^{\mathbf{L}}_B C \to L_{C/A} \to L_{C/B} \to L_{B/A} \otimes^{\mathbf{L}}_B C[1]$$

in $D(C)$.

To this end, we will prove it under certain assumptions, and then reduce the general case to a case where the assumptions are satisfied.

Lemma 2.3.2. *If B is a free algebra over A then Theorem 2.3.1 holds.*

Proof. Denote by $T_\bullet \to C$ the standard resolution of C over B. Since B is a free algebra over A, we have T_n a free algebra over A for every n. Thus, it can be used to calculate $L_{C/A}$ as well as $L_{B/A}$. Therefore from the short exact sequences

$$0 \to \Omega_{B/A} \otimes_B T_n \to \Omega_{T_n/A} \to \Omega_{T_n/B} \to 0$$

we obtain the following short exact sequence:

$$0 \to \Omega_{B/A} \otimes_B T_\bullet \to L_{C/A} \to L_{C/B} \to 0$$

Notice that $L_{B/A} = \Omega_{B/A}[0]$ by Lemma 2.2.4, and each T_n is flat over B therefore we see that the complex associated to $\Omega_{B/A} \otimes_B T_\bullet$ is precisely $L_{B/A} \otimes^{\mathbf{L}}_B C$. Thus the distinguished triangle associated to this short exact sequence of simplicial modules is the one we wanted to prove exists. □

Lemma 2.3.3. *Let $A \to B \to C$ be ring maps such that $B \to C$ is injective. Denote by $P_\bullet \to B$ the standard resolution of B over A and by $Q_\bullet \to C$ the standard resolution of C over B. Then*

$$\overline{Q}_\bullet := Q_\bullet \otimes_{P_\bullet} B$$

is a resolution of C over B. I.e there is a homotopy equivalence between the constant simplicial set B and the underlying simplicial set of $\overline{Q}_\bullet$. Moreover each $\overline{Q}_n$ is a polynomial algebra over B.

Proof. Applying Proposition 1.5.3 to $C = \Delta$, $S_\bullet$ as the identity functor, $\mathcal{O} = P_\bullet$, $\mathcal{O}' = B$, $F = Q_\bullet$ and noting the remark about simplicial modules, we see that $\overline{Q}_\bullet$ is homotopy equivalent to $\underline{C}$. Moreover, since the map $B \to C$ is injective, the induced map $P_\bullet \to Q_\bullet$ has at each n, Q_n a polynomial algebra over P_n (For example, for $n = 1$ it is Q_1 is $P_1[C \setminus B]$). Consequently, $\overline{Q}_n$ is a polynomial algebra over B.

□

This means we can use $\Omega_{\overline{Q}_\bullet/B} \otimes_{\overline{Q}_\bullet} C$ to calculate the cotangent complex of C over B given $B \to C$ is injective.

Lemma 2.3.4. *If $B \to C$ is injective then Theorem 2.3.1 holds.*

Proof. With notation as in the previous lemma, we have the short exact sequence

$$0 \to \Omega_{P_\bullet/A} \otimes_{P_\bullet} C \to \Omega_{Q_\bullet/A} \otimes_{Q_\bullet} C \to \Omega_{\overline{Q}_\bullet/A} \otimes_{\overline{Q}_\bullet} C \to 0$$

Deduced by the short exact sequences

$$0 \to \Omega_{P_n/A} \otimes_{P_n} Q_n \to \Omega_{Q_n/A} \to \Omega_{Q_n/P_n} \to 0$$

and the exactness of the restriction of scalars from Q_n to C. In this case, as $B \to C$ is injective the simplicial module on the right calculates $L_{C/B}$ by the previous lemma, and taking the $L\pi_!$ gives us the desired distinguished triangle.

□

Lemma 2.3.5. *The natural map $L_{B\times C/A} \to L_{B/A} \oplus L_{C/A}$ is an isomorphism*

Proof. Factoring $A \to A[x] \to B \times C$ where the second map is given by $x \mapsto (1, 0)$ gives us the distinguished triangle

$$L_{A[x]/A} \otimes^{\mathbf{L}}_{A[x]} (B \times C) \to L_{B\times C/A} \to L_{B\times C/A[x]} \to L_{A[x]/A} \otimes^{\mathbf{L}}_{A[x]} (B \times C)[1]$$

Similarly we have

$$L_{A[x]/A} \otimes^{\mathbf{L}}_{A[x]} (B) \to L_{B\times C/A} \to L_{B/A[x]} \to L_{A[x]/A} \otimes^{\mathbf{L}}_{A[x]} (B)[1]$$

$$L_{A[x]/A} \otimes^{\mathbf{L}}_{A[x]} (C) \to L_{C/A} \to L_{C/A[x]} \to L_{A[x]/A} \otimes^{\mathbf{L}}_{A[x]} (C)[1]$$

Thus, taking their direct sum, we see it will suffice to prove the case of $B \otimes C$ over $A[x]$. Base changing with the flat map $A[x] \to A[x, x^{-1}]$, we see that $(B \times C) \otimes_{A[x]} A[x, x^{-1}]$ is just $B \otimes_{A[x]} A[x, x^{-1}]$ and $C \otimes_{A[x]} A[x, x^{-1}] = 0$ (since multiplication by (1,0) becomes invertible.). Therefore after this base change the map is an isomorphism. Likewise inverting (x-1) instead kills B and only leaves us with C, showing it is again an isomorphism. Since $(x, x - 1) = (1)$ in $A[x]$ the map itself must be an isomorphism. □

Finally we present the proof of Theorem 2.3.1:

Proof. Since $B \to B \times C$ is injective we have the triangle

$$L_{B/A} \otimes^{\mathbf{L}}_B B \times C \to L_{B\times C/A} \to L_{B\times C/B} \to L_{B/A} \otimes^{\mathbf{L}}_B B \times C[1]$$

in $D(B \times C)$. By the previous lemma, this gives us the distinguished triangle

$$L_{B/A} \otimes^{\mathbf{L}}_B B \times C \to L_{B/A} \oplus L_{C/A} \to L_{B/B} \oplus L_{C/B} \to L_{B/A} \otimes^{\mathbf{L}}_B B \times C[1]$$

Base changing by the flat projection map $(B\times C) \to C$ which kills $L_{B/A}$, this gives us the desired distinguished triangle

$$L_{B/A} \otimes^{\mathbf{L}}_B C \to L_{C/A} \to L_{C/B} \to L_{B/A} \otimes^{\mathbf{L}}_B C[1]$$

□

2.4 Smooth morphisms and deformations

Definition 2.4.1. A ring map $A \to B$ is said to be formally smooth if given a map of A-algebras $C/J \to B$ where $J^2 = 0$

$$\begin{tikzcd} B \arrow[r] \arrow[dr, dashed] & C/J \\ A \arrow[u] \arrow[r] & C \arrow[u] \end{tikzcd}$$

there exists a dashed arrow making the diagram commutative. Moreover if this arrow is unique, then the map is called formally étale.

If in addition to being formally smooth (resp. formally étale), the map is finitely presented, then it is called smooth (resp. étale).

The following immediately follow from the definition:

- Composition of two smooth (resp. étale) morphisms are smooth (resp. étale).
- $A \to A[x]$ is a smooth morphism.
- Base change of a smooth morphism is smooth.
- If for a finite index set I, $A \to B_i$ with $i \in I$ are smooth morphisms then so is $A \to \otimes_{A,i} B_i$

Remark 2.4.2. From the definition, it easily follows that étale morphisms $A \to B$ are precisely smooth morphisms with $\Omega^1_{B/A} = 0$ since two different lifts $l_1, l_2 : B \to C$ would give rise to a non-zero derivation $l_1 - l_2 : B \to J$.

Another important property of smooth morphisms is that they are flat. See [Stacks, Tag 01VF].

Given an A-algebra B, an extension of B by a B-module M is a diagram

$$0 \to M \to B' \to B \to 0$$

which is exact for the underlying abelian groups and where M is a square zero ideal of the A-algebra B'. Given arbitrary M one can always construct an extension by putting on $B \oplus M$ the ring structure given by $(b_1, m_1) \cdot (b_2, m_2) \mapsto (b_1 b_2, b_1 m_2 + b_2 m_1)$. This is called the dual numbers over M. Given B_1', B_2' two extensions, a morphism of extensions is an A-algebra morphism $B_1 \to B_2$ inducing the identity on M. Denote by $Exal_A(B, M)$ the set of isomorphims classes of extensions of B by M. One can put the structure of an abelian group on it in the following way. Given representatives B_1, B_2 of two extensions denote by $\bar{P}$ the pushout

$$\begin{tikzcd} \bar{P} \arrow[r] \arrow[d] & B_1 \arrow[d] \\ B_2 \arrow[r] & B \end{tikzcd}$$

Let $P := \bar{P}/\{(m_1, -m_2) \mid m_1, m_2 \in M\}$. In other words, identify two copies of M that come from B_1, B_2. This will be defined as their sum. It is immediate from the construction that choice of representatives do not matter. This will be defined to be their sum. Clearly, $B \oplus M$ the dual numbers is neutral for this operation. Associativity essentially follows from associativity of forming the pullback and taking quotients. The operation is clearly commutative. And inverse of $\phi : B_1 \to B$ is the extension $B_1 \to B$ given by $b_1 \mapsto -\phi(b)$. Thus we have the structure of an abelian group.

When B is formally smooth over A, we actually have an isomorphism

$$0 = Exal_A(B, M) \xrightarrow{\sim} \mathrm{Ext}^1_B(\Omega^1_{B/A}, M)$$

The map and its inverse are described by the below diagrams

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] & B' \arrow[r] \arrow[d, mapsto] & B \arrow[r] & 0 \\
0 \arrow[r] & M \arrow[r] & \Omega_{B'/A} \otimes_{B'} B \arrow[r] & \Omega_{B/A} \arrow[r] & 0
\end{tikzcd}$$

We know that the bottom row is right exact from properties of differentials. However exactness requires further justification. It is true precisely because B is formally smooth. Formal smoothness allows for a section of the projection $B' \to B$. Call it s. Then clearly $s - id_{B'}$ gives a derivation valued in M which retracts the map $M \to \Omega_{B'/A} \otimes_{B'} B$ therefore the bottom is not only short exact but even split exact.

To describe the inverse, we need a small construction. Given an M extension of $\Omega^1_{B/A}$, K, denote by $\bar{K}$ the pullback

$$\begin{tikzcd}
\bar{K} \arrow[r] \arrow[d] & B \oplus K \arrow[d] \\
B \arrow[r] & B \oplus \Omega^1_{B/A}
\end{tikzcd}$$

where $B \oplus K$ is the ring of dual numbers over K. Then the inverse of the map is given by

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] & K \arrow[r] \arrow[d, mapsto] & \Omega_{B/A} \arrow[r] & 0 \\
0 \arrow[r] & M \arrow[r] & \bar{K} \arrow[r] & B \arrow[r] & 0
\end{tikzcd}$$

From the functoriality of the constructions, it is easy to verify these are mutually inverse and they are group homomorphisms.

Remark 2.4.3. It is worthwhile to note that, for the proof to carry through, all we needed was the exactness of the sequences

$$0 \to M \to \Omega^1_{B'/A} \otimes_{B'} B \to \Omega^1_{B/A} \to 0$$

By assuming formal smoothness, we have actually proved something stronger. That for a formally smooth A algebra B, $Exal_A(B, M) = 0$ for all M the sequence above is split exact, thus its image is trivial. Also showing the differentials are projective modules in this case.

Now, if $P_\bullet \to B$ is the standard resolution of B over A for B not necessarily smooth, we have the functorial isomorphisms

$$0 = Exal_A(P_n, M) \cong \mathrm{Ext}^1_{P_n}(\Omega^1_{P_n/A}, M) = 0$$

for every n. Motivated by this observation, make the following definition:

Definition 2.4.4. Define $Exal_A(P_\bullet, M)$ to be the group of simplicial A-algebras $E_\bullet$ such that there exist an exact sequence

$$0 \to K_\bullet \to E_\bullet \to P_\bullet \to 0$$

where levelwise, it is a square zero extension and such that the complex associated to $K_\bullet$ (as a simplicial A-module under the Dold-Kan correspondance) is quasi-isomorphic to $M[0]$. Two extensions are identified if the complexes associated to the middle terms are quasi-isomorphic. More precisely we identify two extensions admitting a diagram:

$$\begin{tikzcd}
0 \arrow[r] & K_\bullet \arrow[r] \arrow[d] & E_\bullet \arrow[r] \arrow[d] & P_\bullet \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & K'_\bullet \arrow[r] & E'_\bullet \arrow[r] & P_\bullet \arrow[r] & 0
\end{tikzcd}$$

where horizontal arrows are quasi-isomorphisms. Then, one can put a group structure on it defined identically to non-simplicial case.

Note that there is a natural map $Exal_A(B, M) \to Exal_A(P_\bullet, M)$ given by $E \mapsto E \times_B P_\bullet$ that is, the simplicial algebra whose n-simplices are $E \times_B P_n$. Conversely, given an extension of $P_\bullet$

$$0 \to N_\bullet \to E_\bullet \to P_\bullet \to 0$$

Where N is quasi-isomorphic to $M[0]$, one can look at H^0 of their corresponding complexes under Dold-Kan correspondance to obtain an extension of B by M. Moreover, as the complex of $E_\bullet$ sits in an exact sequence between two complexes cohomologically concentrated in degree 0, it is also cohomologically concentrated in degree 0. Then, this extension is equivalent to the extension $H^0(E_\bullet) \times_B P_\bullet$. Showing these maps are mutually inverse, therefore there exist an isomorphism

$$Exal_A(B, M) \xrightarrow{\sim} Exal_A(P_\bullet, M).$$

Moreover, it is easy to notice that this isomorphism is natural in both B and M.

Now, notice each P_n is a smooth A algebra, therefore by the previous discussion we have a natural isomorphism $Exal_A(P_\bullet, M) \to \operatorname{Ext}^1_A(\Omega^1_{P_\bullet/A}, M)$. By [8] I.3.3.4.4, there exist a natural isomorphism $\operatorname{Ext}^1_{P_\bullet}(\Omega^1_{P_\bullet/A}, M) \cong \operatorname{Ext}^1(L_{B/A}, M)$. This gives us a very conceptual way of interpreting the fundamental triangle:

Theorem 2.4.5. *Given a sequence of ring maps $A \to B \to C$ and a C-module M, one has the following long exact sequence*

$$\begin{aligned} 0 \to &\operatorname{Der}_B(C, M) \to \operatorname{Der}_A(C, M) \to \operatorname{Der}_A(B, M) \to Exal_B(C, M) \to \\ \to &Exal_A(C, M) \to Exal_A(B, M) \to \operatorname{Ext}^2_C(L_{C/B}, M) \to \cdots \end{aligned}$$

Proof. Apply the cohomological functor $\operatorname{Ext}(\bullet, M)$ to the fundamental distinguished triangle. ☐

This also immediately gives the following very convenient criterion:

Corollary 2.4.6. *If $L_{B/A}$ is cohomologically concentrated in degree 0 and $\Omega^1_{B/A}$ is projective, then B is smooth over A.*

Soon we will also show that the converse holds.

Proposition 2.4.7. *Let $A \to B$ be a flat ring map. If $L_{B\otimes_A B/B} = 0$ then $L_{B/A} = 0$.*

Proof. Since $B \otimes_A B$ is flat over B we have $L_{B/A} \otimes^{\mathbf{L}}_B (B \otimes_A B) = L_{B\otimes_A B/B}$. Now from the triangle of the triple $B \to B \otimes_A B \to B$ we see

$$0 = L_{B\otimes_A B/B} \otimes^{\mathbf{L}}_{(B\otimes_A B)} B = L_{B/A} \otimes^{\mathbf{L}}_B (B \otimes_A B) \otimes^{\mathbf{L}}_{(B\otimes_A B)} B = L_{B/A}$$

☐

Corollary 2.4.8. *If $A \to B$ is étale, then $L_{B/A} = 0$.*

Proof. This is because it is flat and so is its diagonal . As the diagonal is flat [Stacks, Tag 092N], we have $L_{B\otimes B/B} \cong L_{B/B} = 0$ thus previous proposition applies. ☐

Corollary 2.4.9. *Let $A \to B$ be a ring map and T, S multiplicative subsets with image of S contained in T. Then $L_{T^{-1}B/S^{-1}A} \cong L_{B/A} \otimes_B T^{-1}B$.*

Proof. This follows immediately from the distinguished triangle and the facts $L_{T^{-1}B/S^{-1}A} = L_{T^{-1}B/A}$, $B \to T^{-1}B$ is étale. ☐

Theorem 2.4.10. *Let $A \to B$ be a smooth morphism. Then for every prime $\mathfrak{p}$ of B, there exist $f \notin \mathfrak{p}$ such that there is an étale map $A[x_1, \cdots, x_n] \to B[1/f]$.*

Proof. Since $A \to B$ is smooth, it is finitely presented and $\Omega^1_{B/A}$ is projective. Then $\Omega^1_{B/A} \otimes B_{\mathfrak{p}}$ is free. One can then find an f as specified on which it still is free thanks to Nakayama lemma. Define a map $A[x_1, \cdots, x_n] \to B[1/f]$ by mapping the variables to the elements whose differentials form a basis of $\Omega^1_{B[1/f]/A}$. By construction, this map induces an isomorphism for the differentials. Then the cotangent complex triangle associated to $A \to A[x_1, \cdots, x_n] \to B[f]$ lets us immediately see $L_{B[f]/A[x_1,\cdots,x_n]} = 0$ implying it is étale. □

Corollary 2.4.11. *Given a smooth morphism $A \to B$, $L_{B/A} \cong \Omega^1_{B/A}[0]$.*

Proof. This follows easily using previous theorem and compatibility of the cotangent complex with localisation. □

2.5 Cotangent complex of a morphism of ringed spaces

Let C be a site and $\mathcal{A}$ a sheaf of rings on C. Then one also has an adjoint pair, the forgetful functor and free $\mathcal{A}$-Alg functor between $Sh(C)$ and $\mathcal{A} - Alg$. Now given a map of sheaves of rings $\mathcal{A} \to \mathcal{B}$ one defines the standard resolution and the cotangent complex just as in the case of rings.

Lemma 2.5.1. *Let $f : Sh(C) \to Sh(D)$ be a morphism of sites. Then*

$$f^{-1}L_{\mathcal{B}/\mathcal{A}} = L_{f^{-1}\mathcal{B}/f^{-1}\mathcal{A}}$$

Proof. This follows from the commutativity of the diagram below where horizontal arrows stand for the free/forgetful adjoint pair:

$$\begin{tikzcd} \mathcal{A} - Alg \arrow[r, shift left] \arrow[d, "f^{-1}"] & Sh(C) \arrow[l, shift left] \arrow[d, "f^{-1}"] \\ \mathcal{A} - Alg \arrow[r, shift left] & Sh(C). \arrow[l, shift left] \end{tikzcd}$$

□

Corollary 2.5.2. *$H^i(L_{\mathcal{B}/\mathcal{A}})$ is the sheafification of $U \mapsto H^i(L_{\mathcal{B}(U)/\mathcal{A}(U)})$. Similarly $L_{\mathcal{B}/\mathcal{A}}$ is the sheafification of $U \mapsto L_{\mathcal{B}(U)/\mathcal{A}(U)}$.*

Properties of the cotangent complex we have proved in the case of ring maps so far, also hold for morphisms of schemes. The proofs are carried out identically. Statements about presheaves reduce to the case of ring maps by taking sections. Statements about sheaves follow from the case of presheaves by sheafification. This works since the cotangent complex is compatible with sheafification as above corollary shows.

Remark 2.5.3. In particular, Theorem 2.4.5 holds for maps of schemes. For the details see [8] Théorème 1.1.2.3 and (1.2.5.3).

We end the section with the following:

Definition 2.5.4. Given a map of schemes $f : X \to Y$ define $L_{X/Y}$ to be $L_{\mathcal{O}_X/f^{-1}\mathcal{O}_Y}$ in the small Zariski site of X.

Moreover, due to the compatibility of the cotangent complex with localizations, in the case of a scheme map it will be a complex of quasi-coherent sheaves. For the details see [Stacks, Tag 08T1].

2.6 Application to deformation problems

Definition 2.6.1. Given schemes X_0, Y_0, Y with maps $f_0 : X_0 \to Y_0$, $i : Y_0 \to Y$ if there exist a Y-scheme X such that $X_0 \cong X \times_Y Y_0$ then X is called a *lifting* of X_0 over Y. If f_0 is flat or smooth, one can (and often does) require X to be flat or smooth over Y aswell. In that case it is called a flat lifting or a smooth lifting respectively.

Proposition 2.6.2. *Consider a diagram*

$$\begin{tikzcd} & & X \arrow[d, "f"] \\ T_0 \arrow[r, "i"] \arrow[urr, "g_0"] & T \arrow[r] & Y \end{tikzcd}$$

with f smooth and i a thickening of order 1 with ideal I.

(a) There is an obstruction

$$c(g_0) \in \operatorname{Ext}^1(g_0^*\Omega^1_{X/Y}, I)$$

vanishing of which is equivalent to the existence of a Y-morphism $g : T \to X$ extending g_0 (meaning $gi = g_0$).

(b) If such g exists, the set of extensions g of g_0 form an affine space under $\operatorname{Hom}(g_0^*\Omega^1_{X/Y}, I)$

Proof. As $\Omega^1_{X/Y}$ is locally free of finite rank (i.e dualizable) $\operatorname{Ext}^1(g_0^*\Omega^1_{X/Y}, I) = H^1(T_0, G)$ where $G = \underline{Hom}(g_0^*\Omega^1_{X/Y}, I)$ with $\underline{Hom}$ the internal (i.e sheaf) $\underline{Hom}$. Let U be an open subscheme of T and U_0 the corresponding subscheme of T_0. If given two extensions of $g|_{U_A}$, their difference determines a Y derivation of X valued in $I|_{U_0}$ therefore an element of $G|_U$. Now, since f is smooth, we know locally these extensions must exist. Define a sheaf P over T_0 by $P(U_0) :=$ (Set of extensions of $g|_{U_0}$ to U). Then we have seen P is a G-torsor and $c(g_0)$ is the class of this torsor. This implies (b).

Describing this explicitly using Čech cohomology, given an open cover (U_i) of T, and $g_i \in P(U_i)$ we have an association $g_i - g_j$ over $U_i \cap U_j$ to an element of $G(U_i \cap U_j)$. Moreover, clearly these elements satisfy the cocycle condition. Therefore they determine an element $c(g_0)$ of $H^1(T_0, G)$. If it vanishes, this means the cocycle is also a coboundary meaning g_i will glue to give the desired g thus proving (a). □

Proposition 2.6.3. *Suppose $i : Y_0 \to Y$ is a thickening of order 1 with ideal I and $f_0 : X_0 \to Y_0$ a smooth morphism.*

(a) There is an obstruction

$$\omega(f_0) \in \operatorname{Ext}^2(\Omega^1_{X_0/Y_0}, f_0^*I)$$

vanishing of which is equivalent to existence of a smooth lifting of X over Y.

(b) If a smooth lifting exits, the set of isomorphism classes of smooth liftings form an affine space under $\operatorname{Ext}^1(\Omega^1_{X_0/Y_0}, f_0^*I)$ *(If X_1 and X_2 are two smooth liftings, then an isomorphim between them is a Y-isomorphism inducing the identity as the endomorphism of X_0 obtained through the universal property of fiber product).*

Proof. Since X_0/Y_0 is smooth, $L_{X_0/Y_0} = \Omega_{X_0/Y_0}[0]$. Then both statements follow from Ext applied to the fundamental distinguished triangle of cotangent complexes for $X_0 \to Y_0 \to \operatorname{Spec}\mathbb{Z}$ and Theorem 2.4.5 which also holds for schemes by Remark 2.5.3. □

Given a map of schemes $f : X \to Y$, we can define a cochain complex $\Omega^\bullet_{X/Y}$, with $\Omega^i_{X/Y}$ at degree $i > 0$, $\mathcal{O}_X$ at degree 0 and the differential is the unique map satisfying

- $d = d_{X/Y}$ for degree 0
- $d^2 = 0$
- d is a Y-antiderivation of the exterior algebra. i.e it is $f^{-1}\mathcal{O}_Y$ linear and for homogenous a of degree i $d(ab) = da \wedge b + (-1)^i a \wedge db$

This will be called the *de Rham complex* of X/Y and denoted $\Omega^\bullet_{X/Y}$

3 Frobenius morphism and Cartier isomorphism

Throughout this section, let p denote a fixed prime number.

3.1 Definition and a first discussion

Let X, Y be schemes of characteristic p. In otherwords, schemes such that multiplication by p induces the zero morphism on their structure sheaves. The Frobenius endomorphism of a scheme of characteristic p is the morphism which is identity on the underlying topological space and raising to the power p on the structure sheaf. We will denote the Frobenius endomorphism of X by F_X. If $f : X \to Y$ is a morphism then the following diagram commutes

$$\begin{tikzcd} X \arrow[r, "F_X"] \arrow[d, "f"'] & X \arrow[d, "f"] \\ Y \arrow[r, "F_Y"] & Y . \end{tikzcd} \tag{1}$$

We will denote by $X^{(p)}$ the base change of X by F_Y. Then we have the following diagram

$$\begin{tikzcd} X \arrow[dr, dashed, "F_{X/Y}"] \arrow[drr, bend left, "F_X"] \arrow[ddr, bend right] & & \\ & X^{(p)} \arrow[r, "(F_X)_Y"] \arrow[d] & X \arrow[d, "f"] \\ & Y \arrow[r, "F_Y"] & Y . \end{tikzcd}$$

We will call $F_{X/Y}$ the relative Frobenius of X over Y.

When $Y = \operatorname{Spec} A$ and $X = \operatorname{Spec} A[x_1, \cdots, x_n]$ it can be easily seen that $X^{(p)} = \operatorname{Spec} A[x_1, \cdots, x_n]$. The projection $X^{(p)} \to X$ corresponds to the map raising coefficients to the power of p while $F_{X/Y}$ corresponds to the map which raises variables to the power of p.

Lemma 3.1.1. *The Frobenius F_Y is a universal homeomorphism. In other words, on the underlying topological spaces all base changes of F_Y are homeomorphisms.*

Proof. Consider the following cartesian diagram.

$$\begin{tikzcd} X^{(p)} \arrow[r] \arrow[d, "f"'] & X \arrow[d, "f"] \\ Y \arrow[r, "F_Y"] & Y \end{tikzcd}$$

Clearly, this statement is local on Y. Therefore let $Y = \operatorname{Spec} A$ and φ_A the Frobenius. First we will show the lemma for F_Y. For a radical ideal $\mathfrak{r}$, $x \in \varphi_A^{-1}(\mathfrak{r})$ meaning $x^p \in \mathfrak{r}$ implies $x \in \mathfrak{r}$ therefore $\varphi_A^{-1}(\mathfrak{r}) \subset \mathfrak{r}$. Suppose $\mathfrak{p}$ and $\mathfrak{p}'$ are two distinct prime ideals such that $\varphi_A^{-1}(\mathfrak{p}) = \varphi_A^{-1}(\mathfrak{p}')$. Let $x \in \mathfrak{p}$ and $x \notin \mathfrak{p}'$. Then same is true of x^p. Then $x \in \varphi_A^{-1}(\mathfrak{p}')$ implying $x \in \mathfrak{p}'$. But this is a contradiction, thus the map induced on the spectra is injective.

Now consider $\operatorname{Ker} \varphi_A$. Clearly every element in it is nilpotent therefore it is contained in the nilradical. Then $A/\operatorname{Ker} \varphi_A$ has the same spectrum. Now $A/\operatorname{Ker} \varphi_A \hookrightarrow A$ is an integral extension as for any $s \in A$ its p-th power s^p is in $A/\operatorname{Ker} \varphi_A$. Therefore the induced map on the spectra is surjective. And since under this bijection we clearly have that $D(f)$ for an element corresponds to $D(f^p)$, it is a homeomorphism.

Note that the above proof only uses two facts:
(a) For every $x \in A$ there exist n such that $x^n \in \operatorname{im} \varphi_A$

(b) The kernel is contained in the nilradical.
We will show these properties also hold for its base changes, thus proving the statement.

Let $f : A \to B$ be a morphism and inspect the base change $p_B : B \to B \otimes_A A$. Any element of $B \otimes_A A$ is a finite sum of the form $\sum_i b_i \otimes a_i$. As $a_i b_i^p \mapsto a_i b_i^p \otimes 1$ and $\sum_i b_i^p \otimes a_i^p = (\sum_i a_i b_i^p) \otimes 1$ it follows that $b^p \in \operatorname{im} p_B$ showing (a) holds. Let $\mathfrak{p}_B$ be a prime ideal in B. Then $f^{-1}(\mathfrak{b})$ is a prime ideal in A. Since the map induced by φ_A on spectra is surjective, there exists another prime ideal $\mathfrak{p}_A$ of A such that $\varphi_A^{-1}(\mathfrak{a}) = f^{-1}(\mathfrak{p}_B)$. Then there exist $\mathfrak{p}_{A\otimes B}$ a prime laying over $\mathfrak{p}_A, \mathfrak{p}_B$ meaning $p_B^{-1}(\mathfrak{p}_{A\otimes B}) = \mathfrak{p}_B$ thus the map on the spectra is surjective. Now notice that $\operatorname{im}(\operatorname{Spec} A \otimes B \to \operatorname{Spec} B) \subset V(\operatorname{Ker} p_B)$ therefore (b) holds. This concludes the proof of the lemma. □

Notice, as both the projection $X^{(p)} \to X$ and F_X are homeomorphisms, so is $F_{X/Y}$.

Theorem 3.1.2. *Let Y be a scheme of characteristic p and $f : X \to Y$ a smooth morphism of pure relative dimension n. Then the relative Frobenius $F_{X/Y} : X \to X^{(p)}$ is a finite and flat morphism, and as an $\mathcal{O}_{X^{(p)}} -$ algebra $F_{X/Y_*}\mathcal{O}_X$ is locally free of rank p^n.*

Proof. If $n = 0$ i.e if f is étale, then so is the structure morphism of $X^{(p)}$. As $F_{X/Y}$ is sandwiched between two étale morphisms, the distinguished triangle of cotangent complexes will imply that it is also étale. Since $F_{X/Y}$ is both étale and radicial (it is injective on the topological spaces and the field extensions at the points are purely inseperable) it is an open immersion ([6] 17.9.1). Since it is surjective on the topological spaces, it is an isomorphism. If $n > 0$ locally, f factors through an étale morphism to $\mathbb{A}^n_Y$ and then the projection to Y. Since the desired properties can be checked locally on X and Y we may just assume they are affine and that f factors like this globally. This gives rise to the following diagram

$$\begin{tikzcd}
X \arrow[r, "F_{X/Y}"] \arrow[d] & X^{(p)} \arrow[r] \arrow[d] & X \arrow[d] \\
\mathbb{A}^n_Y \arrow[r, "F_{\mathbb{A}^n_Y/Y}"] \arrow[dr] & \mathbb{A}^n_Y \arrow[r] \arrow[d] & \mathbb{A}^n_Y \arrow[d] \\
& Y \arrow[r, "F_Y"] & Y
\end{tikzcd}$$

This already shows $F_{X/Y}$ to be flat since it is (locally) étale by the argument from the case $n = 0$. Now notice that as the bottom right square and the rectangle with short sides $X^{(p)} \to X,\ Y \xrightarrow{F_Y} Y$ are cartesian squares. This implies the top right square is also cartesian. Since X is étale over $\mathbb{A}^n_Y$, by the argument from the case $n = 0$, the upper rectangle forming the following diagram is cartesian.

$$\begin{tikzcd}
X \arrow[r, "F_X"] \arrow[d] & X \arrow[d] \\
\mathbb{A}^n_Y \arrow[r, "F_{\mathbb{A}^n_Y}"] & \mathbb{A}^n_Y
\end{tikzcd}$$

As the top right square and the upper rectangle are cartesian, so is the top left square. As being finite is stable under base change, it will suffice to prove the statement for $F_{\mathbb{A}^n_Y/Y}$. If $Y = \operatorname{Spec} A$ then $\mathbb{A}^n_Y = \operatorname{Spec} A[x_1, \cdots, x_n]$ and as discussed before $F_{\mathbb{A}^n_Y/Y}$ is the morphism corresponding to taking the p-th power of the variables. Then there is a basis

$$\{x_1^{i_1} \cdots x_n^{i_n} \mid 0 \le i_1 \le p-1, \cdots, 0 \le i_n \le p-1\}$$

□

Denote by s a local section of $\mathcal{O}_X$ and by d the map $d_{X/Y}$. The observation $d(s^p) = ps^{p-1}ds = 0$ implies the following:

$$d(s^p) = F_X^*(ds) = F_{X/Y}^*(1 \otimes ds) = 0$$

Therefore we see that

$$F_X^*\Omega^1_{X/Y} \to \Omega^1_{X/Y}, \quad F_{X/Y}^*\Omega^1_{X^{(p)}/Y} \to \Omega^1_{X/Y}$$

are zero morphisms. Furthermore the differential of $F_{X/Y*}\Omega^\bullet_{X/Y}$ is $\mathcal{O}_{X^{(p)}}$-linear. To prove this, it is sufficent to show $\mathcal{O}_{X^{(p)}}$ linearity of $d = d_{X/Y}$. Furthermore, as there is an isomorphism $\mathcal{O}_{X^{(p)}} \cong F_{X/Y}^*\mathcal{O}_X$ it is sufficent to show linearity for pure tensors under this isomorphism as they additively generate $\mathcal{O}_{X^{(p)}}$. Let $x \otimes y \in \mathcal{O}_{X^{(p)}}$ be a pure tensor. We calculate

$$d((y\otimes x)\cdot s) = d(F_{X/Y}(y\otimes x)s) = d(f(y)x^p s) = f(y)(x^p d(s)+sd(x^p)) = f(y)x^p d(s) = F_{X/Y}(y\otimes x)d(s) = (y\otimes x)\cdot d(s),$$

showing the $\mathcal{O}_{X^{(p)}}$ linearity. Thus, wedge product gives $\bigoplus Z^i F_{X/Y*}\Omega^\bullet_{X/Y}$ (and by projection to $\bigoplus \mathcal{H}^i F_{X/Y*}\Omega^\bullet_{X/Y}$) an anti-commutative $\mathcal{O}_{X^{(p)}}$ algebra structure. The following theorem tells us how this structure of $H^i F_{X/Y*}\Omega^\bullet_{X/Y}$ is related to the one on $\Omega^\bullet_{X^{(p)}/Y}$.

3.2 Cartier isomorphism

Theorem 3.2.1. *Let Y be a scheme of characteristic p and $f : X \to Y$ a morphism.*

(a) There exist a unique homomorphism of graded $\mathcal{O}_{X^{(p)}}$-algebras

$$\gamma : \bigoplus \Omega^i_{X^{(p)}/Y} \to \bigoplus \mathcal{H}^i F_{X/Y*}\Omega^\bullet_{X/Y}$$

with:

- *For $i = 0$, γ is $F_{X/Y}^* : \mathcal{O}_{X^{(p)}} \to F_{X/Y*}\mathcal{O}_X$, which is the morphism associated to the map of schemes $F_{X/Y} : X \to X^{(p)}$*
- *For $i = 1$, γ maps $1 \otimes ds$ to the class of $s^{p-1}ds$, where $1 \otimes ds \in \Omega^1_{X^{(p)}/Y}$ is the element corresponding to $ds \in \Omega^1_{X/Y}$ under the isomorphism $(F_X)_Y^*\Omega^1_X \cong \Omega^1_{X^{(p)}}$.*

(b) If f is smooth, the γ is an isomorphism.

Proof. First we begin by constructing the morphism. We know that $\bigoplus \Omega^i_{X^{(p)}/Y} \cong \bigoplus (F_Y)_X^*\Omega^i_{X/Y}$. Taking into account the fact $(F_Y)_X \circ F_{X/Y} = F_X$, constructing γ is equivalent to constructing a morphism

$$\gamma_{\text{abs}} : \bigoplus \Omega^i_{X/Y} \to \bigoplus \mathcal{H}^i F_{X*}\Omega^\bullet_{X/Y}$$

Consider the map $\mathcal{O}_X \to \mathcal{H}^1 F_{X*}\Omega^\bullet_{X/Y}$ given on the local sections by $s \mapsto [s^{p-1}ds]$. Notice that

$$(s+t)^{p-1}d(s+t) - s^{p-1}ds - t^{p-1}dt = d\left(\frac{(s+t)^p - s^p - t^p}{p}\right)$$

where $\frac{(s+t)^p - s^p - t^p}{p}$ is well defined since each coefficent in the expansion is divisible by p by the binomial theorem. Therefore the map we have constructed is additive. Also note that

$$st \mapsto s^{p-1}t^{p-1}d(st) = t^p s^{p-1}ds + s^p t^{p-1}dt = s \cdot (t^{p-1}dt) + t \cdot (s^{p-1}ds)$$

since considering $\mathcal{H}^i F_{X*}\Omega^\bullet_{X/Y}$ as an $\mathcal{O}_X$-module over through F_{X*} means a local section s of $\mathcal{O}_X$ acts as the action of s^p on $\mathcal{H}^i\Omega^\bullet_{X/Y}$. This shows the map we constructed to be a Y-derivation of $\mathcal{O}_X$. We define $(\gamma)^1_{\text{abs}}$ as the corresponding map from $\Omega^1_{X/Y}$. It is clear from the construction that it satisfies the claimed property. Then as the algebra structure on $\bigoplus \mathcal{H}^i F_{X*}\Omega^\bullet_{X/Y}$ comes from the wedge product inherited from the complex, every element has square zero. Therefore the above map given on degree $0,1$ extends uniquely to all degrees thus proving (a).

Now, if f is smooth then locally it factors as an étale morphism g and the projection h below. As our question is local, replacing X by a suitable affine open we may assume this factorization is global.

$$X \xrightarrow{g} \mathbb{A}^n_Y \xrightarrow{h} Y$$

It was shown in the proof of Theorem 3.1.2 the following diagram is cartesian

$$\begin{tikzcd} X \arrow[r, "F_{X/Y}"] \arrow[d, "g"'] & X^{(p)} \arrow[d, "g^{(p)}"] \\ \mathbb{A}^n_Y \arrow[r, "F_{\mathbb{A}^n_Y/Y}"] & \mathbb{A}^{n\,(p)}_Y \end{tikzcd}$$

$g^*\Omega^i_{\mathbb{A}^n_Y/Y} \to \Omega^i_{X/Y}$ and $g^{(p)\,*}\Omega^i_{\mathbb{A}^{n\,(p)}_Y/Y} \to \Omega^i_{X^{(p)}/Y}$ are isomorphisms as $g, g^{(p)}$ are étale. Since $F_{\mathbb{A}^n_Y/Y}$ is affine and the above diagram is cartesian, the following map is also an isomorphism:

$$g^{(p)\,*}F_{\mathbb{A}^n_Y/Y*}\Omega^i_{\mathbb{A}^n_Y/Y} \to F_{X/Y*}\Omega^i_{X/Y}.$$

As $g^{(p)}$ is étale and consequently flat this isomorphism holds for the cohomology as well

$$g^{(p)\,*}\mathcal{H}^i F_{\mathbb{A}^n_Y/Y*}\Omega^i_{\mathbb{A}^n_Y/Y} \to \mathcal{H}^i F_{X/Y*}\Omega^i_{X/Y}$$

From the construction of γ it is easy to see that this implies the following:
If (b) holds for $\mathbb{A}^n_Y/Y$ it will hold for X. We will now argue that we can further reduce the problem to the case where $Y = \operatorname{Spec}\mathbb{F}_p$. By a slight abuse of notation, for a scheme X over $\mathbb{F}_p$, $\pi : X \to \operatorname{Spec}\mathbb{F}_p$, we will denote $\mathcal{O}_X \otimes_{\pi^{-1}\mathcal{O}_{\operatorname{Spec}\mathbb{F}_p}} -$ by $\mathcal{O}_X \otimes_{\mathbb{F}_p} -$. From the explicit description of $\Omega^i_{\mathbb{A}^n_Y/Y}$ it is clear that $F_{\mathbb{A}^n_Y/Y*}\Omega^\bullet_{\mathbb{A}^n_Y/Y} \cong \mathcal{O}_{Y^{(p)}} \otimes_{\mathbb{F}_p} F_{\mathbb{A}^n_{\mathbb{F}_p}/\mathbb{F}_p}\Omega^\bullet_{\mathbb{A}^n_{\mathbb{F}_p}/\mathbb{F}_p}$, moreover as the differential is $\mathcal{O}_{Y^{(p)}}$-linear we have

$$\mathcal{H}^i F_{\mathbb{A}^n_Y/Y*}\Omega^\bullet_{\mathbb{A}^n_Y/Y} \cong \mathcal{H}^i(\mathcal{O}_{Y^{(p)}} \otimes_{\mathbb{F}_p} F_{\mathbb{A}^n_{\mathbb{F}_p}/\mathbb{F}_p}\Omega^\bullet_{\mathbb{A}^n_{\mathbb{F}_p}/\mathbb{F}_p}) \cong \mathcal{O}_{Y^{(p)}} \otimes_{\mathbb{F}_p} \mathcal{H}^i(F_{\mathbb{A}^n_{\mathbb{F}_p}/\mathbb{F}_p}\Omega^\bullet_{\mathbb{A}^n_{\mathbb{F}_p}/\mathbb{F}_p})$$

It remains to verify (b) for $Y = \operatorname{Spec}\mathbb{F}_p$, $\mathbb{A}^n_Y = \mathbb{F}_p[x_1, \cdots, x_n]$. Then we have to show

$$\begin{aligned} \mathcal{H}^0 F_{\mathbb{A}^n_{\mathbb{F}_p}/\mathbb{F}_p*}\Omega^\bullet_{\mathbb{A}^n_{\mathbb{F}_p}/\mathbb{F}_p} &\cong \mathbb{F}_p[x_1^p, \cdots, x_n^p] \\ \mathcal{H}^1 F_{\mathbb{A}^n_{\mathbb{F}_p}/\mathbb{F}_p*}\Omega^\bullet_{\mathbb{A}^n_{\mathbb{F}_p}/\mathbb{F}_p} &\cong \bigoplus_{1\le i\le n} (x_i^{p-1}dx_i) \cdot \mathbb{F}_p[x_1^p, \cdots, x_n^p] \\ \mathcal{H}^i F_{\mathbb{A}^n_{\mathbb{F}_p}/\mathbb{F}_p*}\Omega^\bullet_{\mathbb{A}^n_{\mathbb{F}_p}/\mathbb{F}_p} &\cong \bigwedge_{1\le i\le n} \mathcal{H}^1 F_{\mathbb{A}^n_{\mathbb{F}_p}/\mathbb{F}_p*}\Omega^\bullet_{\mathbb{A}^n_{\mathbb{F}_p}/\mathbb{F}_p} \end{aligned}$$

Again, from the explicit description of $\Omega^\bullet_{\mathbb{A}^n_{\mathbb{F}_p}/\mathbb{F}_p}$ it is easy to see that $\Omega^\bullet_{\mathbb{A}^n_{\mathbb{F}_p}/\mathbb{F}_p} \cong \bigotimes^n_{\mathbb{F}_p} \Omega^\bullet_{\mathbb{A}^1_{\mathbb{F}_p}/\mathbb{F}_p}$. Using the Künneth formula ([10] Theorem 3.6.3) we see that it will suffice to prove:

$$\begin{aligned} \mathcal{H}^0 F_{\mathbb{A}^1_{\mathbb{F}_p}/\mathbb{F}_p*}\Omega^\bullet_{\mathbb{A}^1_{\mathbb{F}_p}/\mathbb{F}_p} &\cong 1 \cdot \mathbb{F}_p[x^p] \\ \mathcal{H}^1 F_{\mathbb{A}^1_{\mathbb{F}_p}/\mathbb{F}_p*}\Omega^\bullet_{\mathbb{A}^1_{\mathbb{F}_p}/\mathbb{F}_p} &\cong x_1^{p-1}dx \cdot \mathbb{F}_p[x^p] \\ \mathcal{H}^i F_{\mathbb{A}^1_{\mathbb{F}_p}/\mathbb{F}_p*}\Omega^\bullet_{\mathbb{A}^1_{\mathbb{F}_p}/\mathbb{F}_p} &\cong \bigwedge_{1\le i\le n} \mathcal{H}^1 F_{\mathbb{A}^1_{\mathbb{F}_p}/\mathbb{F}_p*}\Omega^\bullet_{\mathbb{A}^1_{\mathbb{F}_p}/\mathbb{F}_p}. \end{aligned}$$

The third one is immediate as it is equivalent to the fact for $i \ge 2$, $\mathcal{H}^i F_{\mathbb{A}^1_{\mathbb{F}_p}/\mathbb{F}_p*}\Omega^\bullet_{\mathbb{A}^1_{\mathbb{F}_p}/\mathbb{F}_p} = 0$. Let t denote the variable of the polynomial ring. $1, t, \cdots, t^{p-1}$ form a basis of $F_{\mathbb{A}^1_{\mathbb{F}_p}/\mathbb{F}_p*}\Omega^\bullet_{\mathbb{A}^1_{\mathbb{F}_p}/\mathbb{F}_p}$ over $\mathbb{F}_p$, as noted during the proof of Theorem 3.1.2. Then from degree 0 to degree 1, the differential $d_0 : F_{\mathbb{A}^1_{\mathbb{F}_p}/\mathbb{F}_p *}\mathbb{A}^1_{\mathbb{F}_p} \to F_{\mathbb{A}^1_{\mathbb{F}_p}/\mathbb{F}_p *}\Omega^1_{\mathbb{A}^1_{\mathbb{F}_p}/\mathbb{F}_p} \cong F_{\mathbb{A}^1_{\mathbb{F}_p}/\mathbb{F}_p *}\mathbb{A}^1_{\mathbb{F}_p} \cdot dt$ can be described by $t^i \mapsto it^{i-1}dt$. Then clearly $\ker d_0$ is spanned by t^p proving the first statement as $t \cdot 1 = t^p$.

Now for the second statement notice that $d(\frac{1}{i}t^i) = t^{i-1}dt$ are in the image for $i \neq p$, However $t^{p-1}dt$ (and any of its multiples) cannot be as it would lead to a contradiction as follows. If $dp(t) = g(t)t^{p-1}dt$ with $g(t)$ a polynomial which has powers of p for exponents of variables with nonzero coefficients. But $\frac{\partial p(t)}{\partial t} = g(t)t^{p-1}$ therefore $p(t) = g(t)t^p + c$ for $c \in \mathbb{F}_p$ implying $dp(t) = 0$ as all variables are raised to a power of p, which is a contradiction. Therefore the second statement also holds and we prove the theorem. ☐

In the second case where γ is an isomorphism, it is denoted by C^{-1} and called the *Cartier isomorphism.* This isomorphism arises naturally from a lifting of the Frobenius morphism over $\mathbb{Z}/p^2\mathbb{Z}$. In the following we examine how.

Let $i : \operatorname{Spec}\mathbb{F}_p \to \operatorname{Spec}\mathbb{Z}/p^2\mathbb{Z}$ be the thickening with the ideal generated by p. Let Y_0 be a scheme of characteristic p and Y a flat lifting of Y_0 over $\operatorname{Spec}\mathbb{Z}/p^2\mathbb{Z}$. Since it is a flat lift, the ideal of $Y_0 \to Y$ is $p\mathcal{O}_Y$. Furthermore since multiplication by p gives an isomorphism

$$\mathbb{F}_p \xrightarrow{\sim} p(\mathbb{Z}/p^2\mathbb{Z})$$

flatness thus implies, after pulling back by the morphism $Y \to \operatorname{Spec}\mathbb{Z}/p^2\mathbb{Z}$, we have an isomorphism

$$\mathbf{p} : \mathcal{O}_{Y_0} \xrightarrow{\sim} p\mathcal{O}_Y$$

Suppose $f_0 : X_0 \to Y_0$ is a smooth morphism and X a smooth lifting of X_0 over Y. Likewise, let X' denote a lifting of $X_0^{(p)}$. If there exist a morphism $F : X \to X'$ such that

$$\begin{tikzcd} X_0 \arrow[r] \arrow[d, "F_{X_0/Y_0}"'] & X \arrow[d, "F"] \\ X_0^{(p)} \arrow[r] & X' \end{tikzcd} \tag{2}$$

commutes then we say F lifts F_{X_0/Y_0}.

Theorem 3.2.2. *With notation as above,*

(a) multiplication by p induces an isomorphism

$$\boldsymbol{p} : \Omega^1_{X_0/Y_0} \xrightarrow{\sim} p\Omega^1_{X/Y}$$

(b) image of

$$F^* : \Omega^1_{X'/Y} \to F_*\Omega^1_{X/Y}$$

is contained in $pF_*\Omega^1_{X/Y}$

(c) Denote by $\varphi_F : \Omega^1_{X_0^{(p)}/Y_0} \to F_{0*}\Omega^1_{X_0/Y_0}$ the homomorphism obtained by "dividing F^* by p". In other words, the unique homomorphism making the square below commutative

$$\begin{tikzcd} \Omega^1_{X'/Y} \arrow[r, "F^*"] \arrow[d] & pF_{0*}\Omega^1_{X/Y} \\ \Omega^1_{X_0^{(p)}/Y_0} \arrow[r, "\varphi_F"] & F_{0*}\Omega^1_{X_0/Y_0}. \arrow[u, "\boldsymbol{p}"'] \end{tikzcd}$$

The image of φ_F is contained in $Z^1F_{0*}\Omega^\bullet_{X_0/Y_0}$ and projecting to the homology gives the Cartier isomorphism in degree 1.

Proof. (a) This follows from the fact $\Omega^1_{X/Y}$ is a flat $\mathcal{O}_X$ module.

(b) The diagram (2) induces the following

$$\begin{array}{ccc} \Omega^1_{X'/Y} & \xrightarrow{F^*} & F_*\Omega^1_{X/Y} \\ \downarrow & & \downarrow \\ \Omega^1_{X_0^{(p)}/Y_0} & \xrightarrow{F^*_{X_0/Y_0}} & F_{X_0/Y_0*}\Omega^1_{X_0/Y_0} \end{array}$$

However by previous discussion we know that the bottom horizontal map is 0. Therefore image of $\Omega^1_{X'/Y}$ is in the kernel of $F_*\Omega^1_{X/Y} \to F_{X_0/Y_0*}\Omega^1_{X_0/Y_0}$.

Now, all we have to show is that the kernel is contained in $pF_*\Omega^1_{X/Y}$. This follows from the fact $\Omega^1_{X/Y} \otimes \mathbb{F}_p \cong \Omega^1_{X_0/Y_0}$.

(c) Suppose a is a local section of $\mathcal{O}_X$ with reduction a_0 modulo p (i.e with image a_0 in $\mathcal{O}_{X_0}$ under the projection map). Denote by $a_0^{(p)}$ the image of a_0 in $\mathcal{O}_{X_0^{(p)}}$ and a' in $\mathcal{O}_{X'}$ a lift of $a_0^{(p)}$. Then

$$F^*a' = a^p + pb$$

for some local section b of $\mathcal{O}_X$ since reduction modulo p of F^*a' is $F^*_{X_0/Y_0}a_0^{(p)} = a_0^p$. Therefore

$$F^*(d') = pa^{p-1}da + pdb$$

finally showing

$$\varphi_F(da_0^{(p)}) = a_0^{p-1}da_0 + db_0$$

where b_0 is the image of b in $\mathcal{O}_{X_0}$ which from the characterization of Cartier isomorphism immediately gives us the result.

□

One of the cases that is going to be particularly important for us is the case of a perfect $\mathbb{F}_p$-algebra A. In that case $W(A)$, the ring of Witt vectors, is a flat lift of A over $\mathbb{Z}_p$ (as it is torsion free over a discrete valuation ring). This implies that $W_2(A)$ is a flat lift of A over $\mathbb{Z}/p^2\mathbb{Z}$. Furthermore, Lemma 6.5.13 of [3] uses the fact relative Frobenius induces the zero morphism in differentials to show a statement which immediately implies that the cotangent complex $L_{A/\mathbb{F}_p} = 0$. Therefore it is even the *unique* flat lift.

4 Decompositon of the de Rham complex

Let k be a field and consider a k-scheme $f : X \to \operatorname{Spec} k$. We define

$$H^i_{\mathrm{DR}}(X/k) := H^i(X, \Omega^\bullet_{X/k}) = \Gamma(\operatorname{Spec} k, R^if_*(\Omega^\bullet_{X/k}))$$

as the i-th de Rham cohomology group of X/k. It is a k-vector space. The first spectral sequence of hypercohomology of $\Gamma(X, \bullet)$ with respect to $\Omega^\bullet_{X/k}$ is called *the Hodge to de Rham spectral sequence of X/k.*

$$E_1^{ij} = H^j(X, \Omega^i_{X/k}) \Rightarrow H^*_{\mathrm{DR}}(X/k)$$

$H^j(X, \Omega^i_{X/k})$ are called *the Hodge cohomology groups of X/k.*

If X is proper over k, the finiteness theorem of Serre-Grothendieck ([5] III 3) implies that the Hodge cohomology groups are finite dimensional as $\Omega^i_{X/k}$ are coherent. Then, Hodge to de Rham spectral sequence implies that

$$\sum_{i+j=n} \dim_k H^j(X, \Omega^i_{X/k}) \geq \dim_k H^n_{\mathrm{DR}}(X/k) \tag{3}$$

and equality is achieved if and only if the spectral sequence degenerates at E_1.

4.1 Degeneration of the Hodge to de Rham spectral sequence in characteristic $p > 0$

Theorem 4.1.1. *Let S be a scheme of characteristic p. Suppose it admits a flat lift T over $\mathbb{Z}/p^2\mathbb{Z}$. Let X be a smooth S scheme. If $X^{(p)}$ admits a smooth lift over T then $\tau_{<p}F_{X/S*}\Omega^\bullet_{X/S}$ is decomposable in $D(X^{(p)})$.*

Proof. As discussed before, giving a decomposition of $\tau_{<p}F_{X/S*}\Omega^\bullet_{X/S}$ is equivalent to giving a morphism in $D(X)$

$$\bigoplus_{i<p}\mathcal{H}^i(F_{X/S}*\Omega^\bullet_{X/S})[-i] \to F_{X/S*}\Omega^\bullet_{X/S}$$

inducing the identity on the cohomology for $i < p$. Taking into account the Cartier isomorphism 3.2.1 this is equivalent to giving

$$\bigoplus_{i<p}\Omega^i_{X^{(p)}/S}[-i] \to F_{X/S*}\Omega^\bullet_{X/S}$$

inducing C^{-1} on the cohomology for $i < p$.

The strategy of the proof will be to produce such an arrow. We do this in three steps.

1)

Proposition 4.1.2. *Suppose $F_{X/S}$ admits a global lift $Z \to Z'$ where Z, Z' are smooth lifts of $X, X^{(p)}$ over T. Define*

$$\varphi_G : \bigoplus_{i<p}\Omega^i_{X^{(p)}/S}[-i] \to F_{X/S*}\Omega^\bullet_{X/S}$$

by

$$\varphi^0_G = F^*_{X/S} : \mathcal{O}_{X^{(p)}} \to F_{X/S*}\mathcal{O}_X \text{ and } \varphi^1_G : \Omega^1_{X^{(p)}/S} \to F_{X/S*}\Omega^1_{X/S}$$

as defined in Theorem 3.2.2 (c). For $i \geq 1$, φ^i_G the composition of $\Lambda^i\varphi^1_G$ with the product $\Lambda^i F_{X/S*}\Omega^1_{X/S} \to F_{X/S*}\Omega^i_{X/S}$. Then φ_G induces C^{-1} on the cohomology for $i < p$.

This follows easily from the fact φ^1_G induces the Cartier isomorphism in degree 1.

2) Here we will obtain from a lift Z' of $X^{(p)}$ over T a decomposition of $\tau_{\leq 1}F_{X/S*}\Omega^\bullet_{X/S}$.

Lemma 4.1.3. *Given two liftings $G_1 : Z_1 \to Z'$, $G_2 : Z_2 \to Z'$ of $F_{X/S}$, we can associate a homomorphism*

$$h(G_1, G_2) : \Omega^1_{X^{(p)}/S} \to F_{X/S*}\mathcal{O}_X$$

such that $\varphi^1_{G^1} - \varphi^1_{G^2} = dh(G_1, G_2)$. If G_3 is another lifting

$$h(G_1, G_2) + h(G_2, G_3) = h(G_1, G_3)$$

Suppose Z_1 and Z_2 are isomorphic as lifts (locally this will be the case). Let $u : Z_1 \xrightarrow{\sim} Z_2$ be an isomorphism. Then G_2u and G_1 lift $F_{X/S}$. Denote by k' the map $X^{(p)} \to Z'$. Proposition 2.6.2 (b) implies G_1 and G_2u differ by a homomorphism $h_u : F^*_{X/S}k'^*\Omega^1_{Z'/T} \to p\mathcal{O}_Z$. Replacing the source and the target using suitable isomorphisms it amounts to a homomorphism $h_u : F^*_{X/S}\Omega^1_{X^{(p)}/S} \to \mathcal{O}_X$. Moreover if v is another isomorphism of Z_1 with Z_2 then by the same arguement u, v differ by a homomorphism $\Omega^1_{X/S} \to \mathcal{O}_X$. Recall that this homomorphism is the one assigning to $dx \in \Omega^1_{X/S}$ the element $u(x) - v(x) \in \mathcal{O}_X$. Similarly, $G_2u - G_2v$ induces a map assigning to $dz \in F^*_{X/S}\Omega^1_{X^{(p)}/S}$ the element $G_2(u(z) - v(z))$. This is the composition of the morphism induced by $u - v$ with the morphism $F^*_{X/S}\Omega^1_{X^{(p)}/S} \to \Omega^1_{X/S}$ induced by reduction of $G_2 \mod p$. As G_2 is a lifting of the relative Frobenius, this reduction is precisely the map induced by the relative Frobenius which we know to be 0. Therefore $G_2 u = G_2 v$. This shows that h_u is independent of u.

Denote by k the inclusion $X \to Z_1$. Consider the diagram below:

$$\begin{tikzcd} & & Z_1 \arrow[d] \\ X \arrow[urr, "k"] \arrow[r] & Z_2 \arrow[r] & T. \end{tikzcd}$$

As $k^*\Omega^1_{Z_1/T} \cong \Omega^1_{X/S}$ the hypotheses of Proposition 2.6.2 are satisfied, implying locally there is a morphism as lifts between Z_1 and Z_2. Any such morphism must be an isomorphism as they are thickenings of $\mathcal{O}_X$ by the same ideal. Therefore they are locally isomorphic. As h_u is independent of u, the homomorphisms obtained locally from the differences of local isomorphisms glue to the desired morphism $h(G_1, G_2)$. That $h(G_1, G_2) + h(G_2, G_3) = h(G_1, G_3)$ holds is clear from the construction and finally $\varphi^1_{G^1} - \varphi^1_{G^2} = dh(G_1, G_2)$ can be seen from the explicit description of φ^1_G at the proof of Theorem 3.2.2.

Given a fixed lifting Z' of $X^{(p)}$ over T, Proposition 2.6.3 implies there is an open cover $\mathcal{U} = (U_i)_{i\in I}$ such that there is a lift Z_i of U_i over T and similarly Proposition 2.6.2 implies there exist a lifting $G_i : Z_i \to Z'$ of $F_{X/S}|_{U_i}$. Identifying the topological spaces of $X, X^{(p)}$, and utilizing Lemma 4.1.3 we have morphims of complexes

$$f_i = \varphi^1_{G_i} : \Omega^1_{X^{(p)}/S}|_{U_i}[-1] \to F_{X/S*}\Omega^\bullet_{X/S}|_{U_i}$$

Furthermore writing $U_{ij} = U_i \cap U_j$, for each pair i, j we have

$$h_{ij} := h(G_i|_{U_{ij}}, G_j|_{U_{ij}}) : \Omega^1_{X^{(p)}/S}|_{U_{ij}} \to F_{X/S*}\Omega^\bullet_{X/S}|_{U_{ij}}$$

The lemma implies that

$$\begin{aligned} f_j - f_i &= dh_{ij} \quad (\text{on } U_{ij}) \\ h_{ij} + h_{jk} &= h_{ik} \quad \text{on the triple intersection} \end{aligned}$$

Thus we can define a morphism of complex of $\mathcal{O}_{X^{(p)}}$ modules

$$\varphi^1_{Z',U,G_i} : \Omega^1_{X^{(p)}/S}[-1] \to \check{\mathcal{C}}(\mathcal{U}, F_{X/S*}\Omega^\bullet_{X/S})$$

Where $\check{\mathcal{C}}(\mathcal{U}, F_{X/S*}\Omega^\bullet_{X/S})$ is the total complex of the Čech bicomplex of $F_{X/S*}\Omega^\bullet_{X/S}$ with respect to the covering $\mathcal{U}$. The components are

$$\check{\mathcal{C}}(\mathcal{U}, F_{X/S*}\Omega^\bullet_{X/S})^n = \bigoplus_{a+b=n} \check{\mathcal{C}}^b(\mathcal{U}, F_{X/S*}\Omega^a_{X/S})$$

The vertical differentials of this bicomplex are the differentials of the de Rham complex and horizontal ones are the usual Čech cohomology differentials (adjusted to make squares anticommutative, to fit into the definition of a bicomplex). In particular

$$\check{\mathcal{C}}(\mathcal{U}, F_{X/S*}\Omega^\bullet_{X/S})^1 = \check{\mathcal{C}}^1(\mathcal{U}, F_{X/S*}\mathcal{O}_X) \oplus \check{\mathcal{C}}^0(\mathcal{U}, F_{X/S*}\Omega^1_{X/S})$$

The morphism $\varphi^1_{G_i}$ in degree 1 is defined as

$$\varphi^1_{G_i}(\omega)(i,j) = (h_{ij}(\omega)|_{U_{ij}}, f_i(\omega)|_{U_i})$$

and 0 elsewhere. Thanks to the relationship of f_i and h_{ij} and the fact h_{ij} satisfy the cocyle condition, this gives a well defined map of complexes. From its construction, it is clear that if one takes a finer cover, the morphism obtained will be compatible given f, h are adjusted accordingly.

Furthermore we have a quasi-isomorphism

$$\epsilon : F_{X/S*}\Omega^\bullet_{X/S} \to \check{\mathcal{C}}(\mathcal{U}, F_{X/S*}\Omega^\bullet_{X/S})$$

as each column is a resolution. Then in the derived category we will compose inverse of this with $\varphi^1_{Z',\mathcal{U},G_i}$ to get the map

$$\varphi^1_{Z'} : \Omega^1_{X^{(p)}/S}[-1] \to F_{X/S_*}\Omega^\bullet_{X/S}$$

Note that if $\mathcal{U}'$ is a refinement of $\mathcal{U}$, with appropriate G'_i then there exist a commutative diagram

$$\begin{tikzcd}
\Omega^1_{X^{(p)}/S}[-1] \arrow[r] \arrow[dr] & \check{\mathcal{C}}(\mathcal{U}, F_{X/S*}\Omega^\bullet_{X/S}) \arrow[d] & F_{X/S*}\Omega^\bullet_{X/S} \arrow[l] \arrow[dl] \\
& \check{\mathcal{C}}(\mathcal{U}', F_{X/S*}\Omega^\bullet_{X/S}), &
\end{tikzcd}$$

showing the arrows obtained in the derived category are identical. Now if there is another cover, one can always find a common refinement showing the construction of the map is independent of a choice of covering.

Finally, to see that $\varphi^1_{Z'}$ induces C^{-1} on the cohomology, notice that we can check it locally which allows us to utilize the lemma from step 1 thus we are done. (Remember that by construction locally $\varphi^1_{Z',\mathcal{U},G_i}$ was the morphism lemma from step 1 dealt with).

3) Now we will get the morphism for the other degrees. Fix again a lifting Z' of $X^{(p)}$. Using the derived tensor product we get

$$(\varphi_{Z'})^{\overset{L}{\otimes} i} : (\Omega^1_{X^{(p)}/S}[-1])^{\overset{L}{\otimes} i} \to (F_{X/S*}\Omega^\bullet_{X/S})^{\overset{L}{\otimes} i}$$

Since both $\Omega^1_{X^{(p)}/S}$ and $F_{X/S*}\Omega^\bullet_{X/S}$ are locally free of finite type (following from the fact $X^{(p)}/S$ is smooth and Theorem 3.1.2 respectively) thus flat, we get:

$$(\Omega^1_{X^{(p)}/S}[-1])^{\overset{L}{\otimes} i} \cong (\Omega^1_{X^{(p)}/S}[-1])^{\otimes i} \qquad (F_{X/S*}\Omega^\bullet_{X/S})^{\overset{L}{\otimes} i} \cong (F_{X/S*}\Omega^\bullet_{X/S})^{\otimes i}$$

For $i < p$, $i!$ is invertible in characteristic p and there exist the following "antisymmetrization" arrow which is a section of the projection of ${\Omega^1_{X/Y}}^{\otimes i}$ to $\Omega^i_{X/Y}$.

$$a : \Omega^i_{X^{(p)}/S}[-i] \to (\Omega^1_{X^{(p)}/S})^{\otimes i}[-i], \quad \omega_1 \wedge \cdots \wedge \omega_i \mapsto \frac{1}{i!}\sum_{\sigma \in S_i} \operatorname{sgn}(\sigma)\omega_{\sigma(1)} \otimes \cdots \otimes \omega_{\sigma(i)}$$

And the product map

$$(F_{X/S*}\Omega^\bullet_{X/S})^{\otimes i} \to F_{X/S*}\Omega^\bullet_{X/S}$$

We define $\varphi^i_{Z'} : \Omega^1_{X^{(p)}/S}[-1]) \to F_{X/S*}\Omega^\bullet_{X/S}$ as the following composite

$$\begin{tikzcd}
(\Omega^1_{X^{(p)}/S})^{\otimes i}[-i] \arrow[r, "(\varphi^1_{Z'})^{\otimes i}"] & (F_{X/S*}\Omega^\bullet_{X/S})^{\otimes i} \arrow[d, "\text{product}"] \\
\Omega^i_{X^{(p)}/S}[-i] \arrow[u, "a"] \arrow[r, "\varphi^i_{Z'}"] & F_{X/S*}\Omega^\bullet_{X/S}.
\end{tikzcd}$$

As antisymmetrization is a section of the projection ${\Omega^1_{X/Y}}^{\otimes i} \to \Omega^i_{X/Y}$ and $\varphi^1_{Z'}$ induces Cartier isomorphism in degree 1, $\varphi^i_{Z'}$ induces C^{-1} over $\mathcal{H}^i$ concluding the proof.

□

Corollary 4.1.4. *Let k be a perfect field of characteristic p and let X be a smooth scheme over $S = \operatorname{Spec} k$. If X is liftable over $T = \operatorname{Spec} W_2(k)$, then $\tau_{\leq p} F_{X/S*}\Omega^\bullet_{X/S}$ is decomposable in $D(X^{(p)})$. If X has dimension less than p then $F_{X/S*}\Omega^\bullet_{X/S}$ is decomposable.*

Corollary 4.1.5. *Let k be a perfect field of characteristic p and let X be a smooth and proper k-scheme of dimension $< p$. If X is liftable over $W_2(k)$, the Hodge to de Rham spectral sequence of X over k degenerates at E_1*

Proof. Since Ω^1 is compatible with change of base and over a field every morphism is flat, the Frobenius isomorphism F_k induces an isomorphism $F_k^* H^j(X, \Omega^i_{X/k}) \xrightarrow{\sim} H^n(X^{(p)}, \Omega^i_{X^{(p)}/k})$. This shows

$$\dim_k H^j(X, \Omega^i_{X/k}) = \dim_k H^j(X^{(p)}, \Omega^i_{X^{(p)}/k})$$

And since $F_{X/k}$ is a homeomorphism, for all n

$$H^n(X^{(p)}, F_{X/k*}\Omega^\bullet_{X/k}) \xrightarrow{\sim} H^n(X, \Omega^\bullet_{X/k}) = H^n_{\mathrm{DR}}(X/k)$$

By assumption, X is liftable over $W_2(k)$ therefore we have a decomposition in $D(X')$

$$\varphi : \oplus \Omega^i_{X^{(p)}/k}[-i] \xrightarrow{\sim} F_{X/k*}\Omega^\bullet_{X/k}$$

Since they are isomorphic in $D(X^{(p)})$, by the functoriality of Cartan-Eilenberg resolutions and the fact that a resolution of the complex in the left hand side has trivial horizontal differentials, we deduce for all n an isomorphism

$$\oplus_{i+j=n} H^j(X^{(p)}, \Omega^i_{X^{(p)},k}) \xrightarrow{\sim} H^n(X^{(p)}, F_{X/k*}\Omega^\bullet_{X/k}).$$

Therefore

$$\sum_{i+j=n} \dim_k H^j(X, \Omega^i_{X/k}) = \dim_k H^n_{\mathrm{DR}}(X/k),$$

Implying the degeneration at sheet 1. □

4.2 Analogue of Kodaira-Akizuki-Nakano vanishing theorem in characteristic $p > 0$

The decomposition obtained in Theorem 4.1.1 allows us to give an easy proof of an analogue of the Kodaira-Akizuki-Nakano vanishing theorem. In the next section, we lift the result to characteristic 0 allowing us to prove the vanishing theorem itself.

Theorem 4.2.1. *Let k be a field of characteristic p and X a smooth projective k-scheme. Let L be an ample invertible sheaf on X. If X is of pure dimension $d < p$ and is liftable over $W_2(k)$. Then following hold*

$$H^j(X, L \otimes \Omega^i_{X/k}) = 0 \quad \text{for } i + j > d$$
$$H^j(X, L^{\otimes -1} \otimes \Omega^i_{X/k}) = 0 \quad \text{for } i + j < d$$

Proof. By Serre duality ([7] III 7.7, 7.12) the vector spaces involved in the formulas are canonically dual. Therefore it is enough to prove either statement. Since L is ample, there exists $n \geq 0$ such that $H^j(X, L^{\otimes p^n} \otimes \Omega^i_{X/k}) = 0$ for all $j > d$ ([7] III 5.2). Therefore if we can prove that for any $n \in N$ this implies $H^j(X, L^{\otimes p^{n-1}} \otimes \Omega^i_{X/k}) = 0$ then we will be done by descending induction. Therefore lets prove the following.

If M is an invertible sheaf over X satisfying $H^j(X, M^{\otimes p} \otimes \Omega^i_{X/k}) = 0$ for all (i, j) such that $i + j > d$, then $H^j(X, M \otimes \Omega^i_{X/k}) = 0$ also for all (i, j) such that $i + j > d$.

Notice that, as M is invertible $F_X^* M \xrightarrow{\sim} M^{\otimes p}$. Recall that F_X denotes the absolute Frobenius. This isomorphism is given by the map $m \mapsto m^{\otimes p}$. Locally, this map is given by $x \otimes 1 \mapsto x^p$, which is an

$\mathcal{O}_{X^{(p)}}$-module isomorphism. Calling $\operatorname{Spec} k = S$ and $(F_X)^*_S M = M'$ we see that this is equivalent to $F^*_{X/Y} M' \xrightarrow{\sim} M^{\otimes p}$. This induces isomorphisms of $\mathcal{O}_{X^{(p)}}$ modules

$$M' \otimes F_{X/S*}\Omega^i_{X/k} \xrightarrow{\sim} F_{X/S*}(F^*_{X/S}M' \otimes \Omega^i_{X/k}) \xrightarrow{\sim} F_{X/S*}(M^{\otimes p} \otimes \Omega^i_{X/k})$$

for all i.

Now consider the first spectral sequence of the hypercohomology of $\Gamma(X^{(p)}, \bullet)$ with respect to the complex $M' \otimes F_{X/S*}\Omega^\bullet_{X/k}$:

$$E^{ij} = H^j(X^{(p)}, M' \otimes F_{X/S*}\Omega^i_{X/k}) \Rightarrow H^*(X^{(p)}, M' \otimes F_{X/S*}\Omega^\bullet_{X/k}).$$

Our hypothesis and the isomorphisms stated above imply that $E_1^{ij} = 0$ for $i + j > d$. We also know that by Corollary 4.1.1 that $F_{X/S*}\Omega^\bullet_{X/k}$ is decomposable. So in $D(X^{(p)})$

$$F_{X/S*}\Omega^\bullet_{X/k} \cong \bigoplus_{i+j=n} \Omega^i_{X^{(p)}/k}[-i]$$

Therefore

$$H^n(X^{(p)}, M' \otimes F_{X/S*}\Omega^\bullet_{X/k}) \cong \bigoplus_{i+j=n} H^j(X^{(p)}, M' \otimes \Omega^i_{X^{(p)}/k})$$

Therefore $H^j(X^{(p)}, M' \otimes \Omega^i_{X^{(p)}/k}) = 0$ for $i + j < d$ and this proves the statement as

$$F^*_S H^j(X, M \otimes \Omega^i_{X/k}) \cong H^j(X^{(p)}, M' \otimes \Omega^i_{X^{(p)}/k})$$

as k-vector spaces.

□

5 Lifting to characteristic 0

In this section, we will try to lift the result obtained in the previous section to characteristic 0.

5.1 The strategy

Let K be a characteristic 0 field and X a smooth proper K-scheme. We will exhibit a diagram

$$\begin{array}{ccccccc} Y & \longrightarrow & Y_1 & \longrightarrow & \mathcal{X} & \longleftarrow & X \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ s & \longrightarrow & \operatorname{Spec} W_2(k) & \longrightarrow & S & \longleftarrow & \operatorname{Spec} K, \end{array}$$

with all squares cartesian, S a $\mathbb{Z}$-algebra of finite type and s a closed point of S. Moreover, Y/s satisfying the assumptions of 4.1.5. Using base change theorems, we will see that this will imply the result for X.

To obtain S as above, we will write K as a filtered colimit of its finitely generated $\mathbb{Z}$-subalgebras. We hope this motivates the following discussion

5.2 Filtered colimits and cartesian systems

Definition 5.2.1. Let $(A_i, u_{ij} : A_i \to A_j (i \leq j)), i, j \in I$ be a filtered colimit of rings and let colim $A_i = A$. Similarly let (E_i, v_{ij}) be a system of A_i modules indexed by I with $\operatorname{colim} E_i = E$. Such a system will be called *cartesian* for every $i \leq j$ the induced morphism

$$u^*_{ij} E_i = A_j \otimes_{A_i} E_i \to E_j$$

is an isomorphism.

In a cartesian system, the following is an isomorphism:

$$u_i^* E_i = A \otimes_{A_i} E_i \xrightarrow{\sim} E.$$

Indeed,

$$E \cong \operatorname{colim}_I E_i \cong \operatorname{colim}_{i \le j} A_j \otimes_{A_i} E_j \cong (\operatorname{colim}_{i \le j} A_j) \otimes_{A_i} E_j \cong A \otimes_{A_i} E_j.$$

If (E_i, v_{ij}) is cartesian, given another system of A_i modules F_i the hom modules $\operatorname{Hom}_{A_i}(E_i, F_i)$ also form an inductive system whose transition maps send $f_i : E_i \to F_i$ to f_j, which is the unique morphism making the diagram below commutative:

$$\begin{tikzcd} A_j \otimes E_i \arrow[r] \arrow[d, "\wr"'] & A_j \otimes F_i \arrow[d] \\ E_j \arrow[r, "f_j"] & F_j \end{tikzcd}$$

Moreover, for each i we can replace the bottom row in the above diagram with $E \to F$ which then gives us a map

$$\operatorname{colim} \operatorname{Hom}_{A_i}(E_i, F_i) \to \operatorname{Hom}(E, F)$$

Given A_i as above; we will be interested in the questions:

(a) If E is a finitely presented A-module, does there exist $i_0 \in I$ and an A_{i_0} module E_{i_0} such that $E \cong u_{i_0}^* E_{i_0}$.

(b) If E_i, F_i are cartesian inductive systems for $i \ge i_0$, with colimits E, F and E_{i_0} is finitely presented, is the map $\operatorname{colim} \operatorname{Hom}_{A_i}(E_i, F_i) \to \operatorname{Hom}_A(E, F)$ an isomorphism?

The answers to both questions are positive and depend on the assumption of finite presentation. For the first question, notice that the generators and relations are a finite subset of A and each element must be a representative of an element of A_j for some j. Then we can take the maximum among these j and call it i_0. This means, generators and relations are representatives of elements of A_i. Thus we can construct a short exact sequence

$$A_{i_0}^n \xrightarrow{\varphi} A_{i_0}^m \longrightarrow \operatorname{Coker}(\varphi) \longrightarrow 0$$

Such that its base change is,

$$A^n \longrightarrow A^m \longrightarrow E \longrightarrow 0$$

Clearly, we are done as $\operatorname{Coker}(\varphi)$ is the E_{i_0} we have been seeking.

We can see the answer to second question similary. The injectivity of the map actually does not need finite presentation. Finite generation is enough. Since E_{i0} is finite and the system is cartesian, so is E. Given a map in the colimit which maps to zero, it can be represented by a diagram of the following form

$$\begin{tikzcd} A \otimes E_i \arrow[r, "\varphi"] \arrow[d, "\wr"'] & A \otimes F_i \arrow[d, "\wr"'] \\ E \arrow[r, "0"] & F. \end{tikzcd}$$

This means images of the generators of E, $\varphi(e_i)$ map to 0 in F. Therefore since the colimit is filtered and there are finite amount of generators we can find $j \ge i$ such that they are already 0 in F_j. Thus, image of ϕ in the colimit is also represented by a map that is 0 meaning ϕ must be zero in the colimit.

Surjectivity on the other hand, depends on the finite presentation. Consider a map

$$f : E \to F.$$

Choose $A^n \to A^m$ such that the cokernel is E. Through composition, f induces a map,

$$A^m \to F$$

which maps image of A^n in A^m maps to 0 in F. This means giving m elements $f_i \in F$ such that they satisfy n equations of the form $\sum_{i=1}^m a_i f_i = 0$. As the colimit is filtered, and there are a finite amount of elements we are concerned with, if we can find such elements in F we can find them in F_j for some $j \geq i_0$ giving us a map

$$A_j^n \to A_j^m \to F_j$$

Clearly, the image of this map in the inductive limit maps to f so we are done.

Now we will examine the situation for schemes. Let $S_i = \operatorname{Spec} A_i$ and denote the maps corresponding to u_{ij} by s_{ji}. Then they form a projective system of schemes, for which $S = \operatorname{Spec} A$ is the projective limit. This essentially is immediate from the antiequivalence between the categories of affine schemes and rings combined with the fact that giving a map to an affine scheme $\operatorname{Spec} A$ from a scheme X is equivalent to giving a ring map $\Gamma(X, \mathcal{O}_X) \to A$. If $(X_i, v_{ij} : X_j \to X_i)$ is a projective system of S_i schemes, it will be called *cartesian* for $i \geq i_0$ if for all $i_0 \leq i \leq j$ the transition arrows make a cartesian square

$$\begin{tikzcd} X_j \arrow[r] \arrow[d] & X_i \arrow[d] \\ S_j \arrow[r] & S_i \end{tikzcd}$$

Note that, since S_i are affine, so are the maps between them. Thus if the system is cartesian for $i \geq i_0$ the scheme $X := X_{i_0} \times_{S_{i_0}} S$ is actually the limit of the projective system (X_i, v_{ij}). This follows from the fact that limits commute with each other and S is the limit of S_i. Now we ask ourself the analogues of the questions we just answered:

(a) If X is an S-scheme of finite presentation, does there exist $i_0 \in I$ and an S_{i_0}-scheme X_{i_0} such that $X \cong S \times_{S_{i_0}} X_{i_0}$.

(b) If X_i, Y_i are cartesian projective systems for $i \geq i_0$, with limits X, Y and X_{i_0} is finitely presented, is the map $\operatorname{colim} \operatorname{Hom}_{S_i}(X_i, Y_i) \to \operatorname{Hom}_S(X, Y)$ an isomorphism?

The strategy is essentially the same as before, pick an affine cover (U_k) of X, and pick an affine cover (V_k^{ij}) for each intersection $U_i \cap U_j$. Since X is of finite presentation, both of these sets can be taken to be finite. Moreover $\Gamma(U_i, \mathcal{O}_X|_{U_i})$ and $\Gamma(V_k^{ij}, \mathcal{O}_X|_{V_{ij}})$ are both finitely presented rings over $\Gamma(S, \mathcal{O}_S)$. With almost an identical argument to before (the only difference is we have algebras here instead of modules) we can find integers j_i and j_k^{ij} such that U_i and V_k^{ij} arise from base change from finitely presented S_{j_i}-, $S_{j_k^{ij}}$-schemes. Since it is a projective system and there are finitely many integers, we can find j smaller than all of them. Call U_i' and ${V_k^{ij}}'$ the schemes obtained by base change to S_j. We can glue U_i' along ${V_k^{ij}}'$. This answers the first question positively. The second question similarly follows by reducing to the affine case and using the fact that we are concerned with finitely many of them.

Moreover, for many properties that are stable under base change, the morphism $X \to S$ might posses, we can find $X_i \to S_i$ with the same property. We will be interested in properness and smoothness.

Lemma 5.2.2. *X, S as above, if $f : X \to S$ has one of the properties: surjective, projective, proper, smooth, then there exist $i_0 \in I$ and a S_{i_0}-scheme X_{i_0} possesing the same property, from which X is induced by base change.*

Proof. We begin by the case where f is projective. In this case we have a closed immersion $i : X \to P_S^n$ for some n given by an ideal I locally of finite type. We will show that we can lift the closed immersion while keeping the ideal locally of finite type. Like before, it will suffice to treat the case X affine since there will be finitely many affines covering X and gluing data among them again given by finitely many affines.

Moreover, since a closed immersion is affine, we may assume the target is affine aswell (more concretely, we can cover P^n_S with standard affine charts and look at their preimages. If we can lift each morphism individually, since there are finitely many we can find an index large enough so that the morphisms glue). Therefore this reduces to the question of, given

$$\Gamma(S, \mathcal{O}_S)[x_1, \dots, x_m] \to \Gamma(S, \mathcal{O}_S)[x_1, \dots, x_n] \to \Gamma(S, \mathcal{O}_S)[x_1, \dots, x_m]/_{(f_1, \dots, f_k)},$$

can we lift it to S_i. By now we are familiar with the arguement that if there are finitely many elements of $\Gamma(S, \mathcal{O}_S)$ involved to construct the sequence, there exist an index such that $\Gamma(S_i, \mathcal{O}_{S_i})$ contains representatives of each of those elements. Therefore we are done.

From now on we will always treat the case X affine. By now it is clear how to proceed from there.

Suppose f is surjective. Lift f to $f_i : X_i \to S_i$ of finite presentation and get a cartesian system for $j \leq i$. Then $f_i(X_i)$ is constructible in S_i. As S_i is affine this means there exist an affine scheme T with a surjective morphism $g_i : T \to S_i \setminus f_i(X_i)$. Morover for any $j \leq i$ it is clear that $f_j(X_j) = s_{ji}^{-1}(f(X_i))$. This shows $T \times_{S_i} S_j$ gives us a cartesian system whose limit has a map $T \times_{S_i} S \to S \setminus f(X)$. However as the right hand side is empty, so is the left hand side. But this can only happen if $T \times_{S_i} S_k$ is empty for some $k \leq i$ because this limit is the spectrum of some directed colimit of rings, which can only be 0 if it is eventually 0. By construction there were surjective maps $T \times_{S_i} S_k \to S_k \setminus f_k(X_k)$. Thus the left handside being empty implies the same for the right side, which is what we wanted to show.

Suppose f is proper. We will use *Chow's Lemma* which can be found in ([6] 8.10.5). Given a proper and finitely presented morphism $f : X \to S$, we can find a commutative diagram

$$\begin{tikzcd} & X' \arrow[dl, "g"'] \arrow[dr, "f'"] & \\ X \arrow[rr, "f"] & & S, \end{tikzcd}$$

with g surjective and f' projective. We have seen we can lift all of these kinds of morphisms. Thus we can find and index i and a commutative diagram

$$\begin{tikzcd} & X'_i \arrow[dl, "g_i"'] \arrow[dr, "f'_i"] & \\ X_i \arrow[rr, "f_i"] & & S_i \end{tikzcd}$$

such that f_i is a lift of f, f'_i is a lift of f' which is also surjective, g'_i a lift of g that is also projective. Now the statement follows from the fact that $f_i \circ f'_i = g_i$ proper and f'_i surjective implies f_i proper.

Now the result will follow from the simple fact if $g \circ f = f'$ is proper and g surjective, this implies f proper.

Suppose f is smooth. We refer to [Stacks, Tag 0C0B]. □

5.3 Finding the closed point

Now, we shift our attention to the closed point s mentioned at the beginning of the section.

Proposition 5.3.1. *Let S be a scheme of finite type over $\mathbb{Z}$. The following hold*

- *(a) If x is a closed point of S, the residue field $k(x)$ is a finite field.*
- *(b) All locally closed nonempty subsets of S contain a closed point of S.*

Note that both properties hold if they hold for each element of a cover $(U_i)_{i \in I}$ of S. This is immediate for property (a). For property (b), suppose there exist an open cover $(U_i)_{i \in I}$ such that for each i property (b) is satisfied for each U_i. Suppose $x \in U_i$ is a closed point of U_i. For (b) to hold for S, it will suffice to show that x is also closed in S. We will show that x is closed in every U_j containing it. Denote by $\overline{\{x\}}$ the closure of x in S . Consider $j \neq i$ such that $\overline{\{x\}} \cap U_j$ is non-empty, in which case $x \in U_j$ necessarily. As U_j also satisfies property (b), there must exist a point $x' \in \overline{\{x\}} \cap U_j$ which is closed in U_j. Suppose $x \neq x'$ and consider $Z = U_j \setminus (U_i \cap U_j)$ which is a closed set of U_j such that $x \notin Z$. Then $x' \notin Z$. However this is a contradiction as this would imply $x' \in U_i$, in which x was closed. Therefore $x = x'$, showing x is closed in S.

Remark 5.3.2. The proof above shows that, for S satisfying property (b), if a point $x \in S$ is closed in an open subset U then it is also closed in S.

From this, we see that it will be enough to treat the affine case. For $S = \operatorname{Spec} R$, we will show that property (b) is equivalent to the property that every radical ideal of R is an intersection of maximal ideals containing it. In other words for a radical ideal $I \subset R$, $I = \text{Jacobson}(I)$. Rings satisfying this property are called Jacobson rings.

Lemma 5.3.3. *Property (b) in Proposition 5.3.1 holds for $S = \operatorname{Spec} R$ if and only if R is Jacobson.*

Proof. Suppose (b) holds. Then for any closed subset $Z \subset \operatorname{Spec} R$, we have $Z = V(I)$ for some radical ideal I. Suppose $I \neq J(I)$. Then there exist $f \notin I$ but $f \in (m)$ for all maximal ideals containing I. Then $D(f) \cap Z \neq \emptyset$ but it cannot contain a maximal ideal by construction which is a contradiction. Therefore for all radical ideals $I = J(I)$.

Suppose R is Jacobson. Take $Z \subset \operatorname{Spec} R$ closed. We need to show for an open U such that $Z \cap U \neq \emptyset$, the intersection contains a maximal ideal. There is no loss of generality in taking $U = D(f)$ for some f. $Z = V(I)$ for some radical ideal I. As R is Jacobson and $f \notin I$ by assumption, there must be a maximal ideal $\mathfrak{m} \supset I$ such that $f \notin \mathfrak{m}$ which is the desired closed point. $\square$

Clearly, quotients and localizations at elements of Jacobson rings are Jacobson.

Now to see an elementary property a non Jacobson ring must satisfy we look at the following

Lemma 5.3.4. *If a ring R is not Jacobson, there exist a prime $\mathfrak{p}$ and an element $f \notin \mathfrak{p}$ such that $\mathfrak{p}$ is not maximal, $V(\mathfrak{p}) \cap D(f) = \{\mathfrak{p}\}$ and $(R/\mathfrak{p})_f$ is a field.*

Proof. Since R is not Jacobson, there must be a locally closed set, nonempty $V(I) \cap D(f)$ such that it does not contain a closed point. Note that this locally closed set is homeomorphic to $\operatorname{Spec}(R/I)_f$. Since any ring has a maximal ideal, there is a point inside corresponding to a prime $\mathfrak{p} \subset R$ which is not maximal. Since it is closed inside $V(I) \cap D(f)$ we must have $V(p) \cap D(f) = \{\mathfrak{p}\}$. Therefore $(R/\mathfrak{p})_f$ is a domain whose spectrum has a single point so it must be a field and we are done. $\square$

Proposition 5.3.5. *If R is Jacobson, and $\varphi : R \to S$ is a morphism finite type then*

(a) S is also Jacobson.

(b) $\operatorname{Spec}(S) \to \operatorname{Spec}(R)$ maps closed points to closed points.

(c) For $\mathfrak{m}' \subset S$ a maximal ideal lying over $\mathfrak{m}$ the field extension $k(\mathfrak{m}')/k(\mathfrak{m})$ is finite.

Proof. Let $\mathfrak{m}' \subset S$ be a maximal ideal and $\varphi^{-1}(\mathfrak{m}') = \mathfrak{m}$. Then notice that $R/\mathfrak{m} \to S/\mathfrak{m}'$ is a ring map of finite type. By [Stacks, Tag 00FH] there exist $f \in R/\mathfrak{m}$ such that $\operatorname{Spec}(R/\mathfrak{m})_f$ is contained in the image of $\operatorname{Spec} S/\mathfrak{m}'$. Therefore $(R/\mathfrak{m})_f$ is a domain whose spectrum consists of a single point. Then it must be a field. By Remark 5.3.2, as $R/\mathfrak{m}$ is Jacobson and (0) is a maximal ideal in $R/\mathfrak{m}_f$, (0) is also a maximal ideal in $R/\mathfrak{m}$. Therefore $R/\mathfrak{m}$ is a field, implying $\mathfrak{m}$ is a maximal ideal in R. Moreover as $S/\mathfrak{m}'$ is finitely generated by $R/\mathfrak{m}$, the Hilbert Nullstellensatz implies $S/\mathfrak{m}'$ is a finite field extension of it. This proves both (b), (c)

Suppose S is not Jacobson. Then there exist a non-maximal prime ideal $\mathfrak{q}$ of S and $g \in S$, $g \notin \mathfrak{q}$ such that $(S/\mathfrak{q})_g$ is a field. Let $\mathfrak{p} = \mathfrak{q} \cap R)$. Then $R/\mathfrak{p} \to (S/\mathfrak{q})_g$ is the situation we looked at above. Therefore $R/\mathfrak{p}$ is a field and $(S/\mathfrak{q})_g$ is an algebraic extension of it. As $S/\mathfrak{q}$ lies in between, it must be a field thus q is maximal giving a contradiction. This proves (a). $\square$

It is easy to notice that $\mathbb{Z}$ is Jacobson, so a scheme S locally of finite type over $\mathbb{Z}$ is also Jacobson. Moreover, the discussion above shows closed points of S get mapped to closed points of $\mathbb{Z}$ and these give rise to algebraic extensions of residue fields. Thus residue fields of the closed points of S are finite extensions of finite fields, thus they are also finite.

5.4 Smooth locus

Proposition 5.4.1. *Let S be an integral scheme finite type over $\mathbb{Z}$. If the generic point ν of S maps to (0) in $\mathbb{Z}$ then the set of points at which S is smooth is open and non-empty.*

Proof. That it is open follows from the fact that, for S to be a smooth around a point x, it is enough to find generators of $\mathcal{O}_{S,x}$ as a $\mathbb{Z}$-module such that their differentials are linearly independent in $\Omega^1_{S/\mathbb{Z},x}$. Choosing a local presentation, this amounts to checking that a determinant of a matrix with polynomial entries does not vanish, which corresponds to a standard open.

To see why it is non-empty, notice that as $\mathbb{Q}$ is perfect, $L_{S_\nu/\operatorname{Spec}\mathbb{Q}}$ is concentrated in degree 0 with projective module of differentials. However this is also equal to $L_{S/\operatorname{Spec}\mathbb{Z}} \otimes_S S_\nu$. Then by Nakayama lemma, one can find an open $U \subset S$ such that $L^1_{U/\operatorname{Spec}\mathbb{Z}} = 0$ and the module of differentials is projective. By [Stacks, Tag 07BU] implies that U is smooth over $\operatorname{Spec}\mathbb{Z}$ and we are done as closed points are dense. □

5.5 Base change

We will state and prove the following result in the case of an affine base scheme S. This hypothesis is actually unnecessary but the proof becomes a bit more involved.

Proposition 5.5.1. *Let S be an affine scheme, Noetherian, integral and $f : X \to S$ a smooth and proper morphism. Then*

(a) $R^j f_*\Omega^i_{X/S}$ and $R^n f_*\Omega^\bullet_{X/S}$ are coherent. Moreover, there exist a non empty open U such that for any i, j, n $R^j f_*\Omega^i_{X/S}|_U$ and $R^n f_*\Omega^\bullet_{X/S}|_U$ are locally free of finite type.

(b) For any i and any morphism $g : S' \to S$ denote by $f' : X' \to S'$ the base change of f by g. Then the canonical base change arrows are isomorphisms in $D(S')$.

$$Lg^* Rf_*\Omega^i_{X/S} \to Rf'_*\Omega^i_{X'/S'} \tag{4}$$

$$Lg^* Rf_*\Omega^\bullet_{X/S} \to Rf'_*\Omega^\bullet_{X'/S'} \tag{5}$$

(c) Given $i \in \mathbb{Z}$, if $R^j f_*\Omega^i_{X/S}$ is locally free over S, of constant rank h^{ij}, then for $j \in \mathbb{Z}$ the base change arrow

$$g^* R^j f_*\Omega^i_{X/S} \to R^j f'_*\Omega^i_{X'/S'}$$

is an isomorphism, implying $R^j f'_*\Omega^i_{X'/S'}$ is also locally free of rank h^{ij}

(d) Same statement as (c) but for $R^n f_*\Omega^\bullet_{X/S}$ instead of $R^j f_*\Omega^i_{X/S}$

Proof.

(a) Since f is proper, $R^j f_*\Omega^i_{X/S}$ coherent by the finiteness theorem of Grothendieck ([5] 3). Notice that, similar to the Hodge to de Rham spectral sequence, there is a spectral sequence

$$E_1^{ij} = R^j f_*\Omega^i_{X/S} \Rightarrow R^n f_*(\Omega^\bullet_{X/S})$$

called the relative Hodge to de Rham spectral sequence. Since our base is Noetherian quotients of coherent modules are coherent. Therefore the $E_\infty^{p,q}$ terms of the spectral sequence will be coherent. Then thanks to the convergenge of the spectral sequence, by a simple induction argument, we can see that $R^n f_*(\Omega^\bullet_{X/S)})$ is also coherent. Denote by $H^{ij} := R^j f_* \Omega^i_{X/S}$ and $H^n := R^n f_*(\Omega^\bullet_{X/S)}$. Let $S = \operatorname{Spec} A$ and K the field of fractions of A. We can obtain K as a filtered colimit of the localizations A_f. Since H^{ij} and H_i are both free of finite type at the generic point, corresponding to K and X is finitely presented over S, then by the machinery we have been developing, there exists an element s such that $H^{ij}|_{D(s)}$, $H^n|_{D(s)}$ must be free of finite type.

(b) Since our morphism is proper, in particular separated, Čech cohomology agrees with sheaf cohomology. This can be seen as follows. Choose a finite covering $\mathcal{U}$ by affines. By base change there exist a covering $\mathcal{U}'$ of X'. As S is affine and X is separated, intersection of affines are again affines. Then by ([7] III 8.7) $f_*\check{C}(\mathcal{U}, \Omega^i_{X/S})$ and $f_*\check{C}(\mathcal{U}', \Omega^i_{X'/S'})$ calculate $Rf_*\Omega^i_{X/S}$ and $Rf_*\Omega^i_{X'/S'}$ respectively. Since Ω^i is compatible with base change we have an isomorphism

$$g^* f_* \check{C}(\mathcal{U}, \Omega^i_{X/S}) \xrightarrow{\sim} f'_* \check{C}(\mathcal{U}', \Omega^i_{X'/S'})$$

As f is smooth, $\Omega^i_{X/Y}$ is locally free therefore flat, so g^* actually computes Lg^* and this isomorphism realizes the claimed isomorphism.

The other argument is identical except, Čech complex is replaced by the total complex of the Čech bicomplex.

(c), (d) We need a lemma

Lemma 5.5.2. *Suppose A is a Noetherian ring and E is a complex of A-modules such that $H^i(E)$ are projective of finite type for all i and nonzero for only finitely many i. Then*

(i) E is isomorphic in $D(A)$ to a bounded complex with projective components of finite type

(ii) If E is bounded with projective components of finite type and B is an A algebra,

$$B \otimes_A H^i(E) \xrightarrow{\sim} H^i(B \otimes E)$$

Proof. For (i), when constructing a projective resolution, for instance the (projective) Cartan-Eilenberg resolution, one starts by giving a surjection to $H^i(E)$, which by projectivity extends to $ker(d_E^i)$ which lies inside E^i. Since $H^i(E)$ is of finite type, there is a surjection from a finite free module onto it. Then, since A is Noetherian, the submodules of this will be finite type aswell. In particular we can keep iterating this to get a free resolution consisting of finite free modules. If we additionally take care to resolve components with vanishing cohomology by the zero modules it is clear that this will produce a bounded complex with finite type components.

For (ii), Notice there must be a largest i such that E^i is non-zero. Let d_E stand for the differentials. Then we have the following exact sequences

$$0 \to im(d_E^{i-1}) \to E^i \to H^i(E) \to 0$$

$$0 \to ker(d_E^{i-1}) \to E^{i-1} \to im(d_E^{i-1}) \to 0$$

$$0 \to im(d_E^{j-1}) \to ker(d_E^j) \to H^j(E) \to 0$$

First one implies $im(d_E^{i-1})$ is projective, following this the second one implies $ker(d_E^{i-1})$ is projective and then third one implies $im(d_E^{i-2})$ is projective. By descending induction we see all the images of diferentials are projective. Therefore, they are flat. This implies for all i

$$0 \to ker(d_E^{i-1}) \otimes B \to E^{i-1} \otimes B \to im(d_E^{i-1}) \otimes B \to 0$$

is exact. We also have for all j

$$0 \to im(d_E^{j-1}) \otimes B \to ker(d_E^j) \otimes B \to H^j(E) \otimes B \to 0$$

exact. Putting these together we see that

$$B \otimes H^i(E) \cong H^i(B \otimes E)$$

□

From this $(c), (d)$ immediately follow. Restrict to an affine of X, $R^n f_* \Omega^\bullet_{X/S}$ and $R^j f_* \Omega^i_{X/S}$ are coherent, and they are cohomologies of $Rf_*\Omega^i_{X/S}$, $Rf_*\Omega^\bullet_{X/S}$ respectively, we land right in the situation of the lemma. By (i) we can pick a complex with projective thus flat components isomorphic to them, so that we don't have to derive g^* and (ii) ensures we can commute taking cohomology and applying Lg^* to reach at the desired result using (b).

□

5.6 Hodge Degeneration Theorem

Theorem 5.6.1. *Let K be a field of characteristic zero and X a smooth and proper K-scheme. Then the Hodge to de Rham spectral sequence of X over K*

$$E_1^{ij} = H^j(X, \Omega^i_{X/K}) \Rightarrow H^*_{\mathrm{DR}}(X/K)$$

degenerates at E_1.

Proof. Let $h^{ij} := \dim_K H^j(X, \Omega^i_{X/K})$ and $h^n = \dim_K H^*_{\mathrm{DR}}(X/K)$. To show the statement, it suffices to show that n $h^n = \sum_{i+j=n} h^{ij}$ for all n.

K is an inductive limit of its sub $\mathbb{Z}$ algebras of finite type. Then, we have seen that there $S = \operatorname{Spec} A$ and a smooth proper S-scheme $\mathcal{X}$ from which X is induced by base change. Moreover, by possibly replacing A by a localization at an element, we may assume that A is smooth over $\operatorname{Spec}\mathbb{Z}$ and the sheaves $R^j f_* \Omega^i_{\mathcal{X}/S}$ and $R^n f_* \Omega^\bullet_{\mathcal{X}/S}$ are free of constant rank. Moreover, by Proposition 5.5.1 (c) and (d), their ranks are necessarily h^{ij} and h^n respectively. Relative dimension of $\mathcal{X}$ over S is a locally constant function because $\mathcal{X}$ is smooth over S. As X is quasi-compact, dimension of X is bounded by an integer d. Let N be the product of all primes smaller than or equal to d. Then, as S is finite type over $\operatorname{Spec}\mathbb{Z}$, there exist a closed point s in $D(N) \subset S$. Then $k(s)$ is a finite field of characteristic $p > d$. Since S is smooth over $\operatorname{Spec}\mathbb{Z}$ the map $k(s) \to S$ extends to a morphism $\operatorname{Spec} W_2(k) \to S$. This data can be presented by the following diagram with cartesian squares

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & Y_1 \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] & X \arrow[l] \arrow[d] \\ s \arrow[r] & \operatorname{Spec} W_2(k) \arrow[r] & S & \operatorname{Spec} K. \arrow[l] \end{tikzcd}$$

Here, Y is a smooth and proper $k(s)$-scheme of dimension smaller than p. Y is liftable over $W_2(k)$, therefore by Theorem 4.1.5, Hodge to de Rham spectral sequence of Y over $k(s)$ degenerates at E_1, implying

$$\sum_{i+j=n} \dim_{k(s)} H^j(Y, \Omega^i_{Y/k(s)}) = \sum_{i+j=n} \dim_{k(s)} H^n_{\mathrm{DR}}(Y/k(s))$$

By (c) and (d) of Proposition 5.5.1,

$$dim_k H^j(Y, \Omega^i_{Y/k(s)}) = h^{ij}, \quad \dim_{k(s)} H^n_{\mathrm{DR}}(Y/k(s)) = h^n$$

proving the degeneration. □

5.7 Kodaira-Akizuki-Nakano Vanishing Theorem

Let K be a field of characteristic zero and X smooth, projective over K of pure dimension d. If L is an ample invertible sheaf on X, then we have

$$H^j(X, L \otimes \Omega^i_{X/K}) = 0 \quad \text{for } i+j > d$$
$$H^j(X, L^{\otimes -1} \otimes \Omega^i_{X/K}) = 0 \quad \text{for } i+j < d$$

Proof. The strategy is identical. However we need to say a bit more about lifting the ample invertible sheaf.

Proposition 5.7.1. *Consider a projective system of affine schemes S_i with limit S and a cartesian system X_i which are S_i schemes of finite presentation for $i \geq i_0$ with limit $X = S \times_{S_{i_0}} X_{i_0}$.*

(a) Given a finitely presented $\mathcal{O}_X$-module E, one can find a finitely presented $\mathcal{O}_{X_i}$-module E_i such that E arises from it by extension of scalars. If X is projective over S and E is ample (resp. very ample), one can find $j \geq i$ such that X_j is projective over S_j and E_j is ample (resp. very ample).

(b) If E_{i_0}, F_{i_0} are finitely presented $\mathcal{O}_{X_{i_0}}$ modules which induce systems $(E_i), (F_i)$ by extension of scalars over X_i for $i \geq i_0$ and E, F which are the extension of scalars over X, then the natural map

$$\lim \operatorname{Hom}_{\mathcal{O}_{X_{i_0}}}(E_i, F_i) \to \operatorname{Hom}_{\mathcal{O}_X}(E, F)$$

is an isomorphism

Proof. Except for the assertion about ampleness, these follow from the affine case by finite presentation arguments. For the ample case, since tensor product commutes with colimits, it is enough to consider the very ample case.

Assume E is very ample. Then there exist a closed immersion $h : X \to \mathbb{P}^r_S$ such that $E \cong h^*\mathcal{O}_{\mathbb{P}^r_S}(1)$. We have seen we can lift projective morphisms and closed immersions, therefore by picking i large enough and utilizing (b), $E \cong h^*\mathcal{O}_{\mathbb{P}^r_S}(1)$ is induced by base change from an isomorphism $E_i \cong h_i^*\mathcal{O}_{\mathbb{P}^r_{S_i}}(1)$. Therefore E is a very ample invertible sheaf. ☐

Now the proof is identical to the Hodge Degeneration theorem. Exhibit the same diagram to find a closed point satisfying the hypothesis of the positive characteristic statement and conclude by Proposition 5.5.1. ☐

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